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Multistep Methods Using Higher Derivatives and Damping at Infinity*

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Abstract. Linear multistep methods using higher derivatives are discussed. The order of damping at infinity which measures the stability behavior of a k-step method for large h is introduced. A-stable methods with positive damping order are most suitable for stiff problems. A method for computing the damping order is given. Necessary and sufficient conditions for A-stability, $A(\alpha)$-stability and stiff stability are presented. A new A-stable two-step method of order 4 with damping order 1 is found and numerical results are given.

1. Introduction. Ordinary linear multistep methods have the drawback that there exist no A-stable methods with an order larger than 2 (see Dahlquist [11]). By introducing higher derivatives, one can break this order barrier, even if one only uses one-step methods (see [12], [13], [14], [16], [19], [20], [25], [28], [29], [30], [37]). In contrast to earlier work (see [12], [16], [23], [24], [25], [26], [35] and [36]), this article is devoted to stability properties for large h. The order of damping at infinity which measures the stability behavior as $\mu h \longrightarrow -\infty$ is introduced. Here μ is an eigenvalue of the Jacobian. This damping order is denoted by ϵ. A method with damping order $\epsilon = 1$ has exactly the property of *damped at* ∞ introduced by Osborne [33]. An A-stable method with positive damping order is stiffly stable defined by Axelsson [2], strongly stable defined by Chipman [7], L-stable defined by Lambert [25]. Enright [16] introduced a class of multistep methods using the second derivative with positive damping order. In Section 3, we give a simple way of computing the order of damping using the so-called Puiseux diagram of a method. Using this result, it is easy to determine which methods are strong candidates for L-stability. The coefficients of some of these methods are listed in the microfiche section of this issue. An A-stable two-step method with damping order $\epsilon = 1$ and error order 4 which uses only $y'(x)$ and $y''(x)$ is given in Section 4. Note that a one-step method with the same error order and stability properties can only be found when $y'''(x)$ is also introduced in the formula. In Section 4 some

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necessary conditions for A-stability are proved; similar conditions are necessary for $A(\alpha)$-stability introduced by Widlund [39] and stiff stability defined by Gear [18]. Moreover, we give a sufficient condition for A-stability. Some numerical examples are presented in Section 5.

2. **Preliminaries.** Consider the multistep method

$$\sum_{i=0}^{k} \alpha_i \mathbf{y}_{n+i} = \sum_{i=0}^{k} \sum_{j=1}^{l_i} h^j \beta_{ij} \mathbf{f}_{n+i}^{(j-1)}, \qquad n = 0, 1, 2, \ldots, \tag{2-1}$$

for solving the initial value problem

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}), \qquad \mathbf{y}(a) = \eta, \qquad x \in I = [a, b), \tag{2-2}$$

where $-\infty < a < b \leqslant +\infty$, $\mathbf{y}$, $\mathbf{f}(x, \mathbf{y}) \in \mathbf{R}^m$. In (2-1) $\mathbf{y}_n$ is an approximation to $\mathbf{y}(x_n)$, $x_n = a + nh$, $h > 0$ and $\mathbf{f}_m^{(j)} = \mathbf{f}^{(j)}(x_m, \mathbf{y}_m)$, where

$$\mathbf{f}^{(0)}(x, \mathbf{y}) = \mathbf{f}(x, \mathbf{y}), \qquad \mathbf{f}^{(j+1)}(x, \mathbf{y}) = \frac{\partial \mathbf{f}^{(j)}(x, \mathbf{y})}{\partial x} + \frac{\partial \mathbf{f}^{(j)}(x, \mathbf{y})}{\partial \mathbf{y}} \mathbf{f}(x, \mathbf{y}),$$

$$j = 0, 1, 2, \ldots.$$

In (2-1) k and l_i, $i = 0(1)k$, are fixed integers and α_i, β_{ij} denote real constants. We shall always assume that $\alpha_k \neq 0$. We say that (2-1) defines a (k, l)-method, where $l := \max_{i=0(1)k} \{l_i\}$. Moreover, we shall always assume that the functions $\mathbf{f}^{(j)}(x, \mathbf{y})$, $j = 0(1)(l-1)$ satisfy a Lipschitz condition in the second variable. If $\Sigma_{j=1}^{l_k} |\beta_{kj}| = 0$, then the method is called explicit, otherwise implicit.

It is convenient to associate with a (k, l)-method the following polynomials

$$\rho(\zeta) := \sum_{i=0}^{k} \alpha_i \zeta^i \quad \text{and} \quad \sigma_j(\zeta) := \sum_{i=0}^{k} \beta_{ij} \zeta^i, \qquad j = 1(1)l,$$

where we have used the definition

$$\beta_{ij} := 0, \qquad j = l_i + 1(1)l.$$

Moreover, we associate with a (k, l)-method the operator

$$L[y, h] := \left(\rho(E) - \sum_{j=1}^{l} \sigma_j(E) h^j D^j \right) y(x), \tag{2-3}$$

where D is the differential operator $Dy(x) = dy(x)/dx$ and E is the shift operator $Ey(x) = y(x + h)$.

Definition 1. The difference operator (2-3) has (error) order p if for all $y \in C_I^{p+1}$

$$L[y, h] = C_{p+1} h^{p+1} y^{(p+1)}(x) + O(h^{p+2}), \tag{2-4}$$

where $C_{p+1} \neq 0$.

According to Dahlquist [9, p. 40], the following assumption can always be made without loss of generality:

(A) The polynomials $\rho(\zeta)$, $\sigma_j(\zeta)$, $j = 1(1)l$ have no common factor (i.e. there is no complex number z such that $\rho(z) = \sigma_j(z) = 0, j = 1(1)l$).

The following theorem has been given by Griepentrog [20] (using a different terminology).

THEOREM 1. (a) *Each of the following conditions are necessary for the convergence of a* (k, l)*-method.*

(i) *The zeros of the polynomial* $\rho(\zeta)$ *lie on the unit disk and the zeros with modulus* 1 *are simple* (*condition of stability*).

(ii) *The associated difference operator* (2-3) *has at least order* 1 (*condition of consistency*).

(b) *A* (k, l)*-method which satisfies* (i) *and* (ii) *is convergent.*

The two following theorems ensure that there exist convergent methods.

THEOREM 2. *To any given real numbers* α_i, $i = 0(1)k$, *with* $\Sigma_{i=0}^{k}\alpha_i = 0$ *and any integers* l_i, $i = 0(1)k$, *there exists a difference operator* (2-3) *with the given* α_i *with an error order* $p \geqslant \Sigma_{i=0}^{k} l_i$.

Theorem 2 can be improved if k is even.

THEOREM 3. *Let* k *be even and let real numbers* α_i, $i = 0(1)k$, *with* $\alpha_i = -\alpha_{k-i}$ *and integers* l_i, $i = 0(1)k$ *with* $l_i = l_{k-i}$ *be given. If* $l_{k/2}$ *is odd, then there exists a difference operator* (2-3) *with the given* α_i *with an order* $p \geqslant q + 1$, *where* $q := \Sigma_{i=0}^{k} l_i$.

Reimer [35] has shown that the error order p of a convergent (k, l)-method satisfies

$$
\text{(2-5)} \qquad \begin{aligned} &p \leqslant l(k+1) + 1 && \text{if } k \text{ even}, l \text{ odd},\\ &p \leqslant l(k+1) && \text{otherwise.} \end{aligned}
$$

Coefficients of methods with $l_i = l$, $i = 0(1)k$ have been computed by Lambert and Mitchell [26] for all cases with $k + l \leqslant 5$. The following $(k, 2)$-methods have been introduced by Enright [16].

Example 1. *Enright's methods.* In these methods $l = l_k = 2$, $l_i = 1$, $i = 0(1)(k-1)$ and $\alpha_k = 1$, $\alpha_{k-1} = -1$, $\alpha_i = 0$, $i = 0(1)k-2$. These methods are given by the formula

$$
\text{(2-6)} \qquad \begin{aligned} y(x+kh) - y(x+(k-1)h) = {} & h\sum_{i=1}^{k}\left(\gamma_i^* - \frac{k+1}{i}\gamma_{k+1}^*\right)\nabla^i y'(x+kh)\\ & + h\gamma_0^* y'(x+kh) + h^2(k+1)\gamma_{k+1}^* y''(x+kh)\\ & + h^{k+3}\left(\gamma_{k+2}^* - \frac{k+1}{k+2}\gamma_{k+1}^*\right) y^{(k+3)}(x+kh)\\ & + O(h^{k+4}), \end{aligned}
$$

where ∇^i is the ith power of the backward difference $\nabla y(x+kh) = y(x+kh) - y(x+(k-1)h)$. The coefficients γ_j^* in (2-6) are defined by

$$
\text{(2-7)} \qquad \gamma_j^* := (-1)^j \int_{-1}^{0} \binom{-s}{j} ds, \qquad j = 0(1)\infty,
$$

(see Henrici [21, p. 194]).

3. The Order of Damping at Infinity. The condition of stability introduced in Section 2 is necessary for the convergence of (k, l)-methods. Hence, it describes the stability as h tends to zero. The following sections are concerned with methods for stiff differential equations, i.e. systems of differential equations where the eigenvalues of the Jacobian lie in the left-hand plane and have real parts which differ greatly in magnitude. In connection with stiff differential equations one usually considers the scalar test initial value problem

$$y' = \mu y, \qquad y(0) = 1, \qquad x \in I = [0, \infty), \tag{3-1}$$

where μ is an arbitrary complex number. When we apply a (k, l)-method to the problem (3-1) then we have the recurrence relation

$$\sum_{i=0}^{k} \alpha_i y_{n+i} - \sum_{i=0}^{k} \sum_{j=1}^{l_i} (h\mu)^j \beta_{ij} y_{n+i} = 0, \qquad n = 0(1)\infty. \tag{3-2}$$

This is a linear homogeneous difference equation with constant coefficients and its characteristic equation is

$$\phi(\zeta, \lambda) = \rho(\zeta) - \sum_{j=1}^{l} \lambda^j \sigma_j(\zeta) = 0, \qquad \lambda := h\mu. \tag{3-3}$$

This is an algebraic equation and hence there exist k solutions $\zeta_i(\lambda)$, $i = 1(1)k$. The function $\zeta_i(\lambda)$ are branches of an algebraic function. Every solution of (3-2) can be written in the form

$$y_n = \sum_{i=1}^{k}{}^{*} \pi_i(n) \zeta_i^n(\lambda), \tag{3-4}$$

where $\pi_i(x)$ is a polynomial of degree less than the multiplicity of the zero $\zeta_i(\lambda)$. The * in the sum of (3-4) indicates that the sum is only taken over those i which have a finite $\zeta_i(\lambda)$. Using (3-4), it is evident that the following lemma holds.

LEMMA 1. *A (k, l)-method is A-stable if and only if* $\max_{j=1(1)k}\{|\zeta_j(\lambda)|\} < 1$ *whenever* $\operatorname{Re} \lambda < 0$.

To measure the behavior of the numerical solution when λ is in a neighborhood of $-\infty$ we introduce the following

Definition 2. A (k, l)-method is damped at infinity of order ϵ if

$$\max_{i=1(1)k} \{|\zeta_i(\lambda)|\} = O\left(\frac{1}{\lambda^\epsilon}\right) \tag{3-5}$$

as λ tends to infinity, that is $|\lambda|^\eta \max_{i=1(1)k}\{|\zeta_i(\lambda)|\}$ is bounded in a neighborhood of ∞ for all $\eta \leqslant \epsilon$.

Osborne [33, p. 201] calls a method with $\epsilon = 1$ "damped at ∞." An A-stable method with a positive ϵ is called stiffly A-stable, Axelsson [2, p. 186], strongly stable, Chipman [7, p. 13]. Lambert [25, p. 236] following Ehle [14, p. 7] calls these methods L-stable. An A-stable method has $\epsilon \geqslant 0$, but $\epsilon \geqslant 0$ does not imply A-stability as can be seen from the following example:

$$y_{n+3} = \frac{1}{11}\{18y_{n+2} - 9y_{n+1} + 2y_n - h6f_{n+3}\}. \tag{3-6}$$

This method has $\epsilon = 1/3$ as we will prove in Theorem 5. Since the error order of (3-6) is 3, it cannot be A-stable (see Dahlquist [11, p. 31]).

Definition 3. Let a (k, l)-method fulfill (A). The set

$$P := \{(i, 0) \mid \text{if } \alpha_i \neq 0\} \cup \{(i, j) \mid \text{if } \beta_{ij} \neq 0\}$$

is called the Puiseux set of the method. The graph of P in R^2 is called the associated Puiseux diagram and is denoted by $G(P)$.

Example 2. The Puiseux diagram $G(P)$ of Enright's (3, 2)-method, described in Example 1, consists of all points in Figure 1 represented by an asterisk.

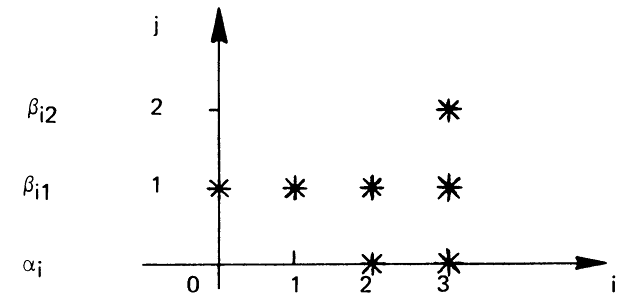


FIGURE 1. *Puiseux diagram of Enright's* (3, 2)*-method*

Without loss of generality we can always assume that

(B) $$\text{if } l_i > 0 \quad \text{then } \beta_{il_i} \neq 0, \ i = 0(1)k.$$

The following theorem provides an easy way to calculate the damping order ϵ.

THEOREM 4. *Let a (k, l)-method fulfill* (A) *and* (B) *and let $G(P)$ be its Puiseux diagram. Then the damping order ϵ is the smallest slope of all straight lines having a finite slope and passing through (k, l_k) and any other point of $G(P)$.*

Proof. We write (3-3) in the form

(3-7) $$\phi(\zeta, \lambda) = \sum_{i=0}^{k} \eta_i(\lambda)\zeta^i = 0,$$

where

(3-8) $$\eta_i(\lambda) := \sum_{j=0}^{l_i} \eta_{ij}\lambda^j := \alpha_i - \sum_{j=1}^{l_i} \beta_{ij}\lambda^j, \qquad i = 0(1)k.$$

It is well known, that (3-7) has k solutions $\zeta_i(\lambda)$, $i = 1(1)k$, which can be represented in a neighborhood of $\lambda = \infty$ by the convergent series

(3-9) $$\zeta_i(\lambda) = e_{1i}\lambda^{-\epsilon_{1i}} + e_{2i}\lambda^{-\epsilon_{2i}} + e_{3i}\lambda^{-\epsilon_{3i}} + \cdots,$$

see, e.g. Ahlfors [1]. Here, $-\infty < \epsilon_{1i} < \epsilon_{2i} < \epsilon_{3i} < \cdots$ and $e_{1i} \neq 0$, $i = 1(1)k$. Hence, we have the damping order

(3-10) $$\epsilon = \min_{i=1(1)k} \{\epsilon_{1i}\}.$$

In the following we compute the smallest ϵ_{1i}. In order to simplify the notation we omit the second index. Since (3-7), we have

$$\begin{aligned}(3\text{-}11)\qquad &\eta_k(\lambda)(e_1\lambda^{-\epsilon_1} + e_2\lambda^{-\epsilon_2} + \cdots)^k + \eta_{k-1}(\lambda)(e_1\lambda^{-\epsilon_1} + e_2\lambda^{-\epsilon_2} + \cdots)^{k-1}\\ &\qquad + \cdots + \eta_1(\lambda)(e_1\lambda^{-\epsilon_1} + e_2\lambda^{-\epsilon_2} + \cdots) + \eta_0(\lambda) \equiv 0.\end{aligned}$$

The series

$$(3\text{-}12)\qquad \zeta(\lambda) = e_1\lambda^{-\epsilon_1} + e_2\lambda^{-\epsilon_2} + e_3\lambda^{-\epsilon_3} + \cdots$$

is absolute convergent in a neighborhood of $\lambda = \infty$, and hence we can expand each $\eta_i(\lambda)\zeta^i(\lambda)$ in (3-11) in a series in powers of λ. Since the sequence ϵ_n, $n = 1(1)\infty$ is strictly increasing, there are terms on the left-hand side of (3-11) with a power of λ which is higher than in all other terms; and the sum over these terms has to vanish. Hence, among the terms

$$\eta_{kl_k}e_1^k\lambda^{l_k - k\epsilon_1},\ \eta_{k-1\,l_{k-1}}e_1^{k-1}\lambda^{l_{k-1}-(k-1)\epsilon_1},\ \ldots,\ \eta_{1l_1}e_1\lambda^{l_1-\epsilon_1},\ \eta_{0l_0}\lambda^{l_0}$$

there are at least two with the same power of λ which is larger than all other powers. This gives us a condition for ϵ_1, and we have therefore by (3-10) to look for the smallest ϵ_1 such that at least two of the numbers

$$(3\text{-}13)\qquad \{l_i - i\epsilon_1 \,|\, \text{if } l_i > 0 \text{ or } \alpha_i \neq 0,\ i = 0(1)k\}$$

are equal and larger than all others. We draw through each point of the set

$$\{(i, j) \in P \,|\, (i, j + t) \notin P,\ t = 1(1)\infty\}$$

a straight line with the slope ϵ_1; see Figure 2.

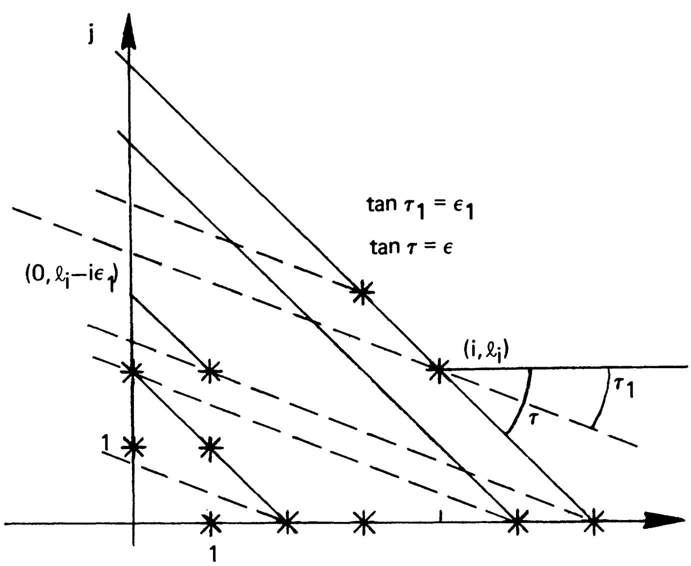


FIGURE 2

Let, for instance, g be the line passing through (i, l_i); then g passes through the vertical axis at the point $(0, l_i - i\epsilon_1)$. Therefore, we have to look for the smallest slope ϵ_1 such that at least two of these lines are identical; and all others have a smaller ordinate at $x = 0$. Clearly, this slope will be ϵ and can be found as described in Theorem 4.

Clearly the following corollary holds.

COROLLARY 1. *Let a* (k, l)*-method fulfill* (A) *and* (B). *Then*

$$\epsilon = \min\left\{ \frac{l_k - l_i}{k - i} \mid \text{if } l_i > 0 \text{ or } \alpha_i \neq 0, i = 0(1)k-1 \right\}.$$

Example 3. We consider Enright's methods described in Example 1. By (2-7) one finds $\gamma_0^* = 1$, $\gamma_j^* < 0$ and $j\gamma_j^* - (j+1)\gamma_{j+1}^* < 0, j = 1(1)\infty$. Hence, $\beta_{01} \neq 0$ and $\beta_{k2} \neq 0$. The Puiseux diagram has, therefore, the form given in Figure 3.

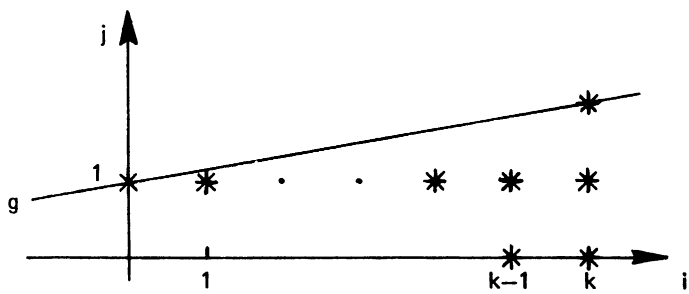


FIGURE 3. *Puiseux diagram of Enright's* $(k, 2)$*-method*

By Theorem 4, ϵ is equal to the slope of the line g in Figure 3, and hence $\epsilon = 1/k$.

Using techniques similar to those of Example 3, it can be easily proved that ϵ has the values given in Table 1. The methods are described in Henrici [21] as well as the different parameters.

Example 4. $(1, l)$*-methods.* For a convergent $(1, l)$-method with (A) and (B) we have by Corollary 1

$$\epsilon = l_1 - l_0. \tag{3-14}$$

This can also be seen directly from (3-7), since

$$\zeta(\lambda) = -\eta_0(\lambda)/\eta_1(\lambda). \tag{3-15}$$

By (3-15), $\zeta(\lambda)$ is a rational function and from (2-4) it follows that

$$-\eta_0(\lambda)/\eta_1(\lambda) = e^\lambda + O(\lambda^{p+1}). \tag{3-16}$$

Here p is the order of the $(1, l)$-method. Hence, each rational approximation to e^λ with $p \geqslant 1$ gives rise to a $(1, l)$-method and vice versa, e.g. let E_{mn} be the (m, n) Padé approximation to e^λ, then the corresponding methods have the following properties, listed in Table 2.

For other $(1, l)$-methods see Davison [13], Liniger, Willoughby [28], Loscalzo [29], Makinson [30], Thompson [37] and for suitable rational approximations to e^λ see Blue, Gummel [4], Cavendish, Culham, Varga [6], Lawson [27], Varga [38].

The proof of the following theorem will be omitted since it follows easily from Theorem 4.

TABLE 1. *Damping order* ϵ of (., 1)*-methods*

Name of method		(., 1)-method	ϵ	error order
Adams-Bashforth		k	-1	k
Adams-Moulton		k	0	k
Nyström		k	-1	k
Milne-Simpson[1]	$q = 0$	1	1	1
	$q = 1$	2	-1	2
	$q = 2$	2	0	4
	$q > 2$	$k = q$	0	$k + 1$
Method based on differentiation at the point x_{n+k-r}	$r = 0$	k	$\frac{1}{k}$	k
	$r = 1, q > 0$	$k = q$	-1	k
	$r > 1, q = 1$	r	$-\frac{1}{k}$	1
Optimal methods (in the sense of Dahlquist)		k	0	$k + 2$

[1] The Milne-Simpson methods are described by the formula $y_n - y_{n-2} = h\Sigma_{m=0}^{q}\kappa_m^*\nabla^m f_n$ where the constants κ_m^* are given in Henrici [21, p. 201].

TABLE 2. *Damping order* ϵ *of* $(1, l)$*-methods corresponding to the Padé approximation*

Method such that $-\eta_0(\lambda)/\eta_1(\lambda) = \cdots$	(1, .)-method	ϵ	error order	A-stability
$E_{l\,l}$	l	0	$2l$	yes 1)
$E_{l\,l-1}$	l	1	$2l - 1$	yes 2)
$E_{l\,l-2}$	l	2	$2l - 2$	yes 2)
$E_{l\,l-3}$	l	3	$2l - 3$	no 2)
$E_{l\,0}, l = 3, 4, 5$	l	l	l	no 3)

1) Birkhoff, Varga [3], 2) Ehle [14], 3) Calahan [5].

THEOREM 5. *The order of damping at infinity of a* (k, l)*-method with* (A), (B) *and* $\eta_0(\lambda) \not\equiv 0$ *satisfies*

(3-17) $$\epsilon \leqslant l/k.$$

In (3-17) *we have equality if and only if*

(3-18a) $$l_k = l, \quad l_0 = 0, \quad \alpha_0 \neq 0$$

and

(3.18b) $$l_i \leqslant \left[i \frac{l}{k}\right] \quad \text{if } \eta_i(\lambda) \not\equiv 0, \ i = 1(1)k-1.^{**}$$

A (k, l)-method which fulfills (A), (B) and (3-18) is called a (k, l)-method with optimal damping order. The $(k, 1)$-methods with optimal ϵ and highest order are the methods based on differentiation with $r = 0$ (see Henrici [21, p. 206]). These methods are only convergent for $k = 1(1)6$ (see Cryer [8]), but for $k \leqslant 6$ they are very efficient for solving stiff differential equations (see Gear [18]). Using techniques given in Jeltsch [24], it can be shown that to any given k, l and ϵ there exists a unique (k, l)-method with damping order ϵ and an error order which is larger than the error order of any other (k, l)-method with damping order ϵ. For $\epsilon > 0$ these methods have been determined by solving the linear system of equations which arises from (2-4) (see, e. g. Lambert and Mitchell [26]). The computation was carried out by a SYMBAL-program.*** The coefficients of the methods together with the error order and C_{p+1} are listed in Tables 3 and 4 for $k = 1, 2, 3$; $l = 1, 2, 3, 4$ and $k = 4$, $l = 1, 2, 3$. The tables appear in the microfiche section of this issue. All except four of the methods are stable and hence convergent. The results in Table 3, together with the results of Dahlquist [10], and Cryer [8] and Reimer [35], seem to suggest that to each (k, l) there might exist a critical $\epsilon_{k,l}$ such that to a given damping order ϵ the (k, l)-method of highest error order is stable if and only if $\epsilon \geqslant \epsilon_{k,l}$.

4. Necessary and Sufficient Conditions for A-Stability. We shall prove the following.

THEOREM 6. *Let a (k, l)-method fulfill* (A) *and* (B). *The following conditions are necessary for A-stability.*

(i) $\epsilon \geqslant 0$.

(ii) $l = l_k$.

(iii) *If $\epsilon = 0$, then the zeros of the polynomial $\sigma_l(\zeta)$ lie on the closed unit disk.*

(iv) *The zeros of $\eta_k(\lambda)$ have a positive real part.*

(v) $\max_{j=1(1)k} \{|\zeta_j(iy)|\} \leqslant 1$ *for all $y \in R$.*

Moreover, (iv) *and* (v) *together are sufficient for A-stability.*

Proof. (i) is a trivial consequence of Definition 2 and Lemma 1, and (ii) follows from (i) using Corollary 1. To show (iii) we look at the leading term $e_1\lambda^{-\epsilon_1}$ of the series (3-12). Since $\epsilon_1 = \epsilon = 0$, the term in (3-11) belonging to the highest power of λ, namely λ^0, can be written as $\sigma_l(e_1)$. This term has to vanish and hence e_1 is a zero of $\sigma_l(\zeta)$. By Lemma 1 and (3-9), (iii) follows immediately. To show (iv) let λ_1 be a zero of $\eta_k(\lambda)$. If $\lambda = \lambda_1$, then (3-7) reduces to $\Sigma_{i=0}^{k-1}\eta_i(\lambda)\zeta^i$ which is a polynomial in

**[a] denotes the largest integer not exceeding a.

***SYMBAL is a formula manipulation language. For a description see Engeli [15]. The computation was carried out on the CDC 6400/6500 of the Swiss Federal Institute of Technology at Zürich.

ζ of degree $k-1$ or less. Hence, one of the k-solutions $\zeta_i(\lambda)$, $i = 1(1)k$ of (3-7) tends to infinity as λ tends to λ_1. This establishes (iv). That (v) is necessary is trivial since by Lemma 1 the algebraic functions $\zeta_j(\lambda)$ can have no poles on $\lambda = iy$, $y \in R$ and hence are continuous in a strip containing the imaginary axis. To prove that (iv) and (v) are sufficient for A-stability we first observe that by (iv) the functions $\zeta_j(\lambda)$ have no poles in the left-hand plane since $\eta_k(\lambda) \neq 0$ whenever Re $\lambda \leqslant 0$. Moreover, since the $\zeta_j(\lambda)$ are algebraic, (v) implies that we have no pole at $\lambda = \infty$; in fact, we have

$$\max_{j=1(1)k} \{|\zeta_j(\infty)|\} \leqslant 1. \tag{4-1}$$

The $\zeta_j(\lambda)$ can be considered as the k branches of an algebraic function on a Riemann surface. By the maximum principle for analytic functions on Riemann surfaces (see, e.g. Pfluger [34, p. 17]) and the continuity of algebraic functions (v) together with (4-1) implies that either $|\zeta_j(\lambda)| < 1, j = 1(1)k$ whenever Re $\lambda < 0$ or $\zeta_j(\lambda) =$ constant. However, $\zeta_j(\lambda) =$ constant implies that $\rho(\zeta)$ and $\sigma_j(\zeta)$, $j = 1(1)l$ have a common factor, and this is a contradiction to (A). Hence, by Lemma 1 the method is A-stable. This completes the proof of Theorem 6.

The condition (ii) in Theorem 6 contains as a special case the result of Dahlquist [11] that an explicit, $(k, 1)$-method cannot be A-stable. Clearly, (i)–(iii) are also necessary conditions for stiff stability defined by Gear [18] and $A(\alpha)$-stability defined by Widlund [39]. In the same way as we proved (iv) in Theorem 6 one can show that the zeros of $\eta_k(\lambda)$ cannot lie in the region of absolute stability $\Omega :=$ $\{\lambda | \max_{i=1(1)k} |\zeta_i(\lambda)| < 1\}$. The conditions (i)–(iv) are not sufficient for A-stability since the $(3, 1)$-method given in (3-6) fulfills (i)–(iv), $\eta_3(\lambda) = 1 - 6\lambda/11$, and is not A-stable. Using Theorem 6, it is easy to show that Enright's $(2, 2)$-method and the following method,

$$17y_{n+2} - 16y_{n+1} - y_n = h(10f_{n+2} + 8f_{n+1}) - 2h^2 f_{n+2}^{(1)} \tag{4-2}$$

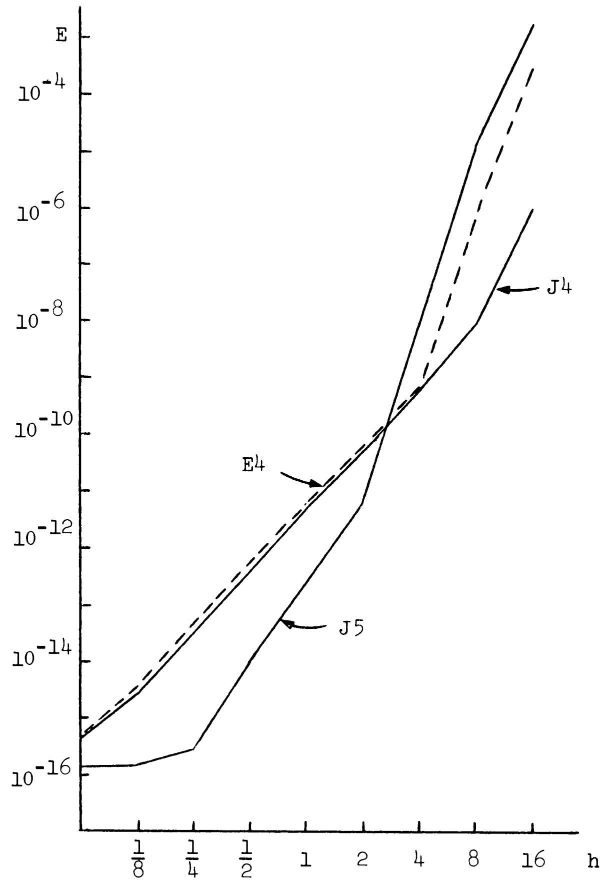
are A-stable. Condition (iv) is trivially satisfied since $(-1)^j \eta_{kj} > 0, j = 0, 1, 2$. Condition (v) can be verified by making the transformation $\zeta = (z + 1)/(z - 1)$, $z = (\zeta + 1)/(\zeta - 1)$ and using a theorem due to Frank (see, e.g. Marden [31, p. 179]). We omit the details since it consists only of tedious computations. For brevity we refer to these methods as E4 and J4. Both are L-stable since the damping order is $\epsilon = 1/2$, $\epsilon = 1$, respectively. The error order of both methods is 4. The author knows no other L-stable $(k, 2)$-method with an order exceeding 3.

5. Numerical Illustration. In the following examples we compare the methods J4, J5 and E4. J5 is given by the formula

$$23y_{n+2} - 16y_{n+1} - 7y_n = h(12f_{n+2} + 16f_{n+1} + 2f_n) - 2h^2 f_{n+2}^{(1)}$$

(see also Tables 3 and 4 of the microfiche section of this issue). J5 is not A-stable since $\phi(-1, \lambda) = 0$ has a root in the left-hand plane. However, it is easily verified that J5 is A_0-stable (i.e. $|\zeta_j(x)| < 1, j = 1, 2$ whenever $x \in (-\infty, 0)$). J5 has error order 5 and $\epsilon = 1/2$.

FIGURE 4. *Relative error E at x = 32 as a function of the stepsize h*



Example 5. Consider the initial value problem proposed by Gear [18]:

$$y' = \lambda(y - g(x)) + g'(x), \qquad y(0) = g(0) + c.$$

The exact solution is $y(x) = g(x) + ce^{\lambda x}$. Here c, λ are arbitrary parameters and $g(x)$ an arbitrary function in $C^1[0, \infty)$. If λ is much smaller than zero and $g(x)$ is smooth, then the problem is stiff. We choose $c = 1$, $\lambda = -20$ and $g(x) = \arctan(x)$. The exact solution is $y(x) = \arctan(x) + e^{-20x}$. The methods are started with exact starting values. At each step the implicit equation (2-1) for y_{n+k} is solved analytically. The numerical results are given in Table 5 and Figure 4 where E_n denotes the relative error, that is $E_n = |(y_n - y(x_n))y(x_n)^{-1}|$. One observes in Figure 4 that for $h < 2$ the methods behave as predicted by the classical theory (see, e.g. Henrici [21]); that is, the higher order method J5 is better than the lower order methods E4 and J4. Moreover, of the two methods of order 4 the one with the smaller error constant

$$C = C_{p+1}/\sigma_1(1)$$

(see Henrici [21, p. 223]) is better. One has $C = 1/270$ for J4 and $C = 7/1440$

TABLE 5. *Relative error E_n*[1] *of* J4, J5 *and* E4 *as a function of x for Example* 5

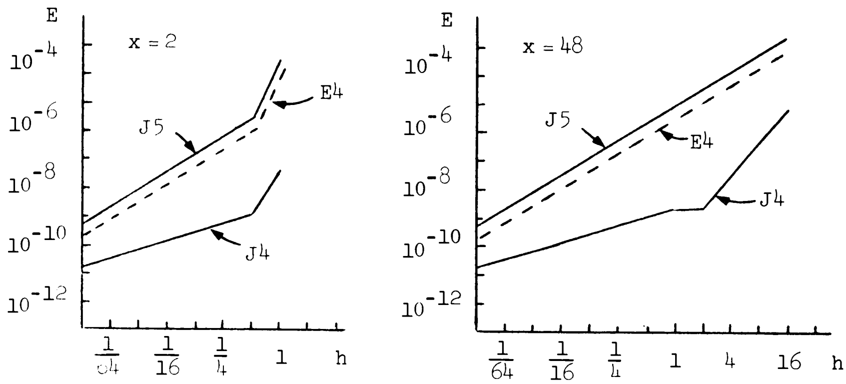
Method		E4		J4		J5	
h		1	2	1	2	1	2
x_n	$y(x_n)$	E_n	E_n	E_n	E_n	E_n	E_n
2	1.107	6.1(-2)	-	9.5(-3)	-	2.8(-1)	-
4	1.326	1.0(-3)	2.8(-2)	1.6(-4)	1.6(-3)	1.2(-2)	1.5(-1)
6	1.406	2.2(-5)	1.9(-3)	2.1(-6)	1.7(-4)	7.4(-6)	2.4(-2)
8	1.446	2.6(-7)	2.1(-4)	1.6(-7)	9.1(-6)	1.0(-5)	1.2(-3)
10	1.447	6.3(-8)	2.3(-5)	5.8(-8)	1.9(-6)	2.2(-7)	2.5(-4)
	⋮						
32	1.540	5.6(-11)	6.0(-10)	4.6(-11)	4.9(-10)	2.4(-12)	5.8(-11)
	⋮						
64	1.555	6.8(-13)	8.2(-12)	6.7(-13)	6.8(-12)	1.6(-14)	3.9(-13)
	⋮						
124	1.5627	1.7(-14)	1.5(-13)	1.4(-14)	1.3(-13)	1.4(-15)	0 machine
126	1.5629	1.6(-14)	1.3(-13)	1.3(-14)	1.1(-13)	1.4(-15)	1.4(-15)
128	1.5630	1.4(-14)	1.2(-13)	1.1(-14)	1.1(-13)	1.4(-15)	0 machine

[1] The integer in parenthesis indicates powers of 10 by which the preceding numbers are to be multiplied.

TABLE 6. *Number of Newton iterations used while integrating with* J4 *from* $x = 0$ *to* $x = 48$ (N = *total number of iterations,* A = $Nh/(48 - h)$ = *average number of iterations per integration step*)

h	2^{-7}	2^{-6}	2^{-5}	2^{-4}	2^{-3}	2^{-2}	2^{-1}	1	2	4	8	16
N	12289	6145	3073	1537	961	573	285	142	70	45	24	13
A	2.0005	2.0009	2.002	2.004	2.5	3.0	3.0	3.02	3.04	4.09	4.8	6.5

FIGURE 5. E = *largest relative error of* $y_{1\,n}$, $y_{2\,n}$ *and* $y_{3\,n}$ *at* $x = 2$ *and* $x = 48$ *as a function of the stepsize h*



for E4. However, for $h > 2$ the order of damping at infinity seems to come into play. This change of behavior occurs in this example only for relatively large h; however, in the following example it occurs for some h smaller than 2^{-7}.

Example 6. The following stiff initial value problem arose from a chemistry problem (see Gear [18]).

$$
(5\text{-}1)\qquad
\begin{aligned}
y_1'(x) &= -0.013y_2(x) - 1000y_1(x)y_2(x) - 2500y_1(x)y_3(x),\\
y_2'(x) &= -0.013y_2(x) - 1000y_1(x)y_2(x),\\
y_3'(x) &= -2500y_1(x)y_3(x),
\end{aligned}
$$

with $y_1(0) = 0, y_2(0) = 1, y_3(0) = 1$. For $x = 2$ and $x = 48$ the exact solution is

$$
(5\text{-}2)\qquad
\begin{aligned}
y_1(2) &= -0.3616933169289\ (-5), & y_1(48) &= -.1945338956808\ (-5),\\
y_2(2) &= 0.9815029948230, & y_2(48) &= 0.6110474831446,\\
y_3(2) &= 1.018493388244, & y_3(48) &= 1.388950571516.
\end{aligned}
$$

These values are correct to 1.5 units of the last given digit. The eigenvalues of the Jacobian (along the exact solution) are

$$\lambda_1(x) \equiv 0,\quad \lambda_2(x) \in [-0.01, 0],\quad \lambda_3(x) \in [-4200, -3499]\quad \text{for } x \in [0, 48].$$

A constant step h is used throughout the integration. The methods are started using the SSP-routine DRKGS with an error tolerance 10^{-14}. The nonlinear equation (2-1) is solved using Newton's method. As first approximation to $y_{i\,n+2}$ the value $y_{i\,n+1}^2/y_{in}$ has been used, where $y_{i\,n}$ corresponds to $y_i(x_n)$. The Newton iteration was terminated as soon as the relative change in each component was less than 10^{-12}. The number of Newton steps was always the same for all three methods. In Table 6 the total number of Newton iterations and the average number of Newton iterations used in one itegration step is given.

In Figure 5 the largest of the relative errors in the three components are given as a function of $h = 2^n$ at $x = 2$ and $x = 48$. One should appreciate the fact that the relative error was always in all components approximately the same despite the fact that $y_1(x)\,10^5 \sim y_2(x)$, $x \in [1, 48]$. For example, we found with J4, $h = 2$

at $x = 48$ the relative errors 2.3×10^{-9}, 1.9×10^{-9} and 0.8×10^{-9} for the first, second and third component, respectively. Observe that the two methods with $\epsilon = ½$ behave in the same way, despite the fact that one is of error order 4 and the other of error order 5. Moreover, J4 with $\epsilon = 1$ behaves differently from E4 with $\epsilon = ½$; but both have error order 4. For example, to get a relative error of at most 10^{-8} at $x = 48$ the number of Newton iterations used by E4 is 20 times larger than the one used by J4. The calculations have been performed on an IBM 360 using double precision (i.e. 56-bit mantissa).

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TABLE I-1

TABLE I: Taylor Series Coefficients $a_{j_1 j_2 j_3}$ of the Jacobian Function sn(u,k). $j_1=(n+1)/2$; $j_1+j_2+j_3=n$.

$a_{j_1 j_2 j_3}$	j_2
n = 17	
1	0
8071256	1
2949965020	2
47152124264	3
109645021894	4
47152124264	5
2949965020	6
8071256	7
1	8
n = 19	
1	0
72641337	1
74197080276	2
2504055894564	3
11966116940238	4
11966116940238	5
2504055894564	6
74197080276	7
72641337	8
1	9
n = 21	
1	0
653772070	1
18595397731885	2
1284534958887560	3
11715171542338290	4
23478363658644484	5
11715171542338290	6
1284534958887560	7
18595397731885	8
653772070	9
1	10

TABLE I-2

n = 23

1	0
5883948671	1
465352380000235	2
6460701405171285	3
10726661133042009O	4
393938089395885894	5
393938089395885894	6
107266611330420090	7
6460701405171285	8
465352380000235	9
5883948671	10
1	11

---

n = 25

1	0
52955538084	1
1163848723925346	2
321298267540551700	3
9412382749388124015	4
59752013018382750024	5
107947764316226205276	6
59752013018382750024	7
9412382749388124015	8
321298267540551700	9
1163848723925346	10
52955538084	11
1	12

---

n = 27

1	0
476599842805	1
29100851707716150	2
15875718186751193446	3
803475280086029066515	4
8470841585571575617239	5
25835579116799316507780	6
25835579116799316507780	7
8470841585571575617239	8
803475280086029066515	9
15875718186751193446	10
29100851707716150	11
476599842805	12
1	13

TABLE I-3

n = 29

1	0
4289398585298	1
7275668079778918O3	2
781562415106660985428	3
67362921649153881472361	4
1146456994425541774291534	5
5632500127524872577252027	6
9424979520638053300516632	7
5632500127524872577252027	8
1146456994425541774291534	9
67362921649153881472361	10
781562415106660985428	11
727566807977891803	12
4289398585298	13
1	14

n = 31

1	0
3860458726773 9	1
181896141522008736 21	2
383965994860847705699 51	3
55811535120723314177812 29	4
150221961163114696686151695	5
114933097355930733743223 5521	6
305180887553895144099052 5939	7
305180887553895144099052 5939	8
114933097355930733743223 5521	9
150221961163114696686151695	10
55811535120723314177812 29	11
383965994860847705699 51	12
181896141522008736 21	13
3860458726773 9	14
1	15

n = 33

1	0
34744128540971 2	1
4547446582165021936 56	2
1884152729554433297404688	3
4588149201749047758262574 36	4
19239380962379456298762250416	5
2235593827697951673190930866664	6
90646772394907350101746588686 4	7
14299533293027343920930446469 82	8
90646772394907350101746588686 4	9
2235593827697951673190930866664	10
19239380962379456298762250416	11
4588149201749047758262574 36	12
1884152729554433297404688	13
4547446582165021936 56	14
34744128540971 2	15
1	16

TABLE I-4

n = 35

1	0
3126971568687473	1
11368657974646161302248	2
92396925087242863212482504	3
37524907781760654616571819884	4
2424371762015227695363084225932	5
41982964485265754951017173213880	6
252583298644057469403578416269848	7
602297594518030428986818986545686	8
602297594518030428986818986545686	9
252583298644057469403578416269848	10
41982964485265754951017173213880	11
2424371762015227695363084225932	12
37524907781760654616571819884	13
92396925087242863212482504	14
11368657974646161302248	15
3126971568687473	16
1	17

n = 37

1	0
28142744118187326	1
284216848055029040209305	2
4529421792220618780953132624	3
3058692313447287528959880082164	4
301977301501927982712251650296648	5
7681155057059283727400087851836804	6
67077985737611839850488056248053296	7
234170438234669757816987374536542702	8
352513571679334580855533139395470836	9
234170438234669757816987374536542702	10
67077985737611839850488056248053296	11
7681155057059283727400087851836804	12
301977301501927982712251650296648	13
3058692313447287528959880082164	14
4529421792220618780953132624	15
284216848055029040209305	16
28142744118187326	17
1	18

n = 39

1	0
253284697063686007	1
7105425014717554019615631	2
221994390052130259394532925609	3
248766472286660081843970414904068	4
37303324488483426954302995423715292	5
1378203273696399945207173716059020652	6
17173078391956624011742130717002163700	7
85635607007228962104291998560813839198	8
187377221472810770345920109207417275058	9
187377221472810770345920109207417275058	10
85635607007228962104291998560813839198	11
17173078391956624011742130717002163700	12
1378203273696399945207173716059020652	13
37303324488483426954302995423715292	14
248766472286660081843970414904068	15
221994390052130259394532925609	16
7105425014717554019615631	17
253284697063686007	18
1	19

TABLE I-5

n = 41

1	0
2279562273573174140	1
177635661714292879129333150	2
10879128434075642651641785959580	3
20203253868959518771392559825506285	4
4580796878616173620118408244176914608	5
243689884438907962985939130480387999720	6
4274452522959262726615911031357902157680	7
29863765165573633115609534911253825271570	8
92446695058285716391958652429341086990280	9
133969576487544409027917496382111749031668	10
92446695058285716391958652429341086990280	11
29863765165573633115609534911253825271570	12
4274452522959262726615911031357902157680	13
243689884438907962985939130480387999720	14
4580796878616173620118408244176914608	15
20203253868959518771392559825506285	16
10879128434075642651641785959580	17
177635661714292879129333150	18
2279562273573174140	19
1	20

n = 43

1	0
20516060462158567341	1
4440891888211006424569211370	2
533114507941647087221696108146570	3
1639243235717722648852313037046453305	4
560128160549135541529462577248201125213	5
42615064610639130927440580694098016933848	6
1040957982950195520590134594765664197681560	7
10033617862597411302371670253845686246467650	8
43017289543421872952097748472726300184788090	9
87728918732198931944006930116683301862240028	10
87728918732198931944006930116683301862240028	11
43017289543421872952097748472726300184788090	12
10033617862597411302371670253845686246467650	13
1040957982950195520590134594765664197681560	14
42615064610639130927440580694098016933848	15
560128160549135541529462577248201125213	16
1639243235717722648852313037046453305	17
533114507941647087221696108146570	18
4440891888211006424569211370	19
20516060462158567341	20
1	21

TABLE I-6

n = 45

1	0
184644544159427106154	1
111022300477586804328521775591	2
26123594546702044085526699031503100	3
132923444218451509189072119405846654355	4
68283013125971106451240903324125753716898	5
7390289780812377609071588455917782806706037	6
249230724309924443773931882861214431260678608	7
3273247158340961478628421988032806047303042330	8
19106900875568186798772740152583987267862674420	9
53584249785424102912573513119405869087185148918	10
75239204631157522675631000601051937933922764392	11
53584249785424102912573513119405869087185148918	12
19106900875568186798772740152583987267862674420	13
3273247158340961478628421988032806047303042330	14
249230724309924443773931882861214431260678608	15
7390289780812377609071588455917782806706037	16
68283013125971106451240903324125753716898	17
132923444218451509189072119405846654355	18
26123594546702044085526699031503100	19
111022300477586804328521775591	20
184644544159427106154	21
1	22

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n = 47

1	0
16618008974348843955475	1
2775557542867631254917084671065	2
1280082056495642083638458387900885491	3
10774309557783598613466933578712446737055	4
8306051633303508006760284701965046518394653	5
1273545261977469819494923690008367374545329695	6
58883620568814372676383312320742625862669958405	7
1043080227289567521787610710182510189123900602538	8
8175785866750908617220315276737911020940067654750	9
30949875721911960369001953290125975409197457599738	10
59521218780319064216950833748175975316172942823870	11
59521218780319064216950833748175975316172942823870	12
30949875721911960369001953290125975409197457599738	13
8175785866750908617220315276737911020940067654750	14
1043080227289567521787610710182510189123900602538	15
58883620568814372676383312320742625862669958405	16
1273545261977469819494923690008367374545329695	17
8306051633303508006760284701965046518394653	18
10774309557783598613466933578712446737055	19
1280082056495642083638458387900885491	20
2775557542867631254917084671065	21
16618008974348843955475	22
1	23

TABLE I-7

n = 49

1	0
1495620807691359559368	1
69388938863336838872742230963028	2
62724702167664030806105463320712967928	3
873107580262755356033649662814680367103746	4
1008798346891106069674845062004217099062126168	5
218418024200718459604948339142805127847887061764	6
13767561449229500950631899040013363589097919031464	7
326218687075211486962131649800062757253999650634479	8
3394443742229202076972991369333038170312239508202448	9
17084515829874885039358166642716434050375758695548328	10
44110314824135914973179189027929995303934228678921264	11
60310604513008106191189250732031934051971919257179164	12
44110314824135914973179189027929995303934228678921264	13
17084515829874885039358166642716434050375758695548328	14
3394443742229202076972991369333038170312239508202448	15
326218687075211486962131649800062757253999650634479	16
13767561449229500950631899040013363589097919031464	17
218418024200718459604948339142805127847887061764	18
1008798346891106069674845062004217099062126168	19
873107580262755356033649662814680367103746	20
62724702167664030806105463320712967928	21
69388938863336838872742230963028	22
1495620807691359559368	23
1	24

TABLE II-1

TABLE II: Taylor Series Coefficients $b_{h_1h_2h_3}$ of the Jacobian Function cn(u,k). The Coefficients $c_{r_1r_2r_3}$ of the Jacobian Function dn(u,k) are given by the relation

$$c_{r_1r_2r_3} = c_{h_1h_3h_2} = b_{h_1h_2h_3};\ h_2=n/2,\ r_1+r_2+r_3=h_1+h_2+h_3=n$$

n = 16

$b_{h_1h_2h_3}$	h_3
1	0
2690416	1
586629984	2
6337665152	3
9860488448	4
2536974336	5
67047424	6
16384	7

n = 18

$b_{h_1h_2h_3}$	h_3
1	0
24213776	1
14804306080	2
345558617984	3
1165333452544	4
782931974144	5
95153582080	6
1073463296	7
65536	8

n = 20

$b_{h_1h_2h_3}$	h_3
1	0
217924020	1
371548371744	2
17992189979904	3
119641512257280	4
171748920960000	5
57102164668416	6
3497455190016	7
17178624000	8
262144	9

TABLE II-2

n = 22

1	0
1961316220	1
930341916 5040	2
912656818686720	3
1128380217174 9120	4
308839837311 49824	5
221717809820 46720	6
395983927345 1520	7
127231162122240	8
274872401920	9
1048576	10

n = 24

1	0
17651846024	1
232733558500720	2
456088324 44953280	3
100905359289141 1200	4
49012294500889559 04	5
66554715390779228 16	6
26320284127739904 00	7
2663009512026931 20	8
460423584022528 0	9
43980223938 56	10
4194304	11

n = 26

1	0
158866614264	1
5819812891661136	2
2259853542156884800	3
87202761117626211840	4
716376110273114701824	5
1693272192484782391296	6
1274464676219028701184	7
295751682605237207040	8
17573163611835596800	9
1661790989278248 96	10
70368639320064	11
16777216	12

TABLE II-3

n = 28

1	0
1429799528428	1
145509858586733712	2
111428311714927846144	3
7370225067120481047040	4
99020093727436106102784	5
385190678334759891320832	6
502186837918249197109248	7
225195504620332576997376	8
32032609142507284725760	9
1146127667313268228096	10
5989977008224862208	11
1125899453857792	12
67108864	13

n = 30

1	0
12868195755908	1
3637888729721421568	2
5479122038971541617408	3
613887494691792209993216	4
13167638443349313860675584	5
81021396112222611059515392	6
171737069430034825499246592	7
133923514868455741736681472	8
37616624304591039218581504	9
3383855154588868186996736	10
74208892231402953637888	11
215770902543537799168	12
18014396563324928	13
268435456	14

n = 32

1	0
115813761803232	1
90948601574079299520	2
268999019240499899029760	3
50643171484933322929049088	4
1704114648317720171436085248	5
16111208467616945884560171008	6
53082196221488518115144663040	7
67083998960526192929674690560	8
33093884724665138608054730752	9
60368894965559854643231588352	10
351400311330636141063831552	11
4783248052841380409507840	12
7770040319983516385280	13
288230367830212608	14
1073741824	15

TABLE II-4

n = 34

1	0
1042323856229152	1
2273728415841470761536	2
13195094474226710295619328	3
4151473793790552383625069056	4
216338605052759260110060220416	5
3075140830049570551391879225344	6
15235746354655353693186036957184	7
29684920174729444761653821243392	8
23795728083567594767162297810944	9
7733649264326537084367655665664	10
941239114566906745772662849536	11
36064401405578205314485846016	12
307456606607989921890172928	13
279760939080040920907776	14
4611685982993907712	15
4294967296	16

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n = 36

1	0
9380914706062436	1
56843339123033013338944	2
646940478992848398769505792	3
338902250192956077259381785088	4
27090381753547809667495429068800	5
569463705621852446817394895577088	6
4137664126710959136708569940623360	7
11979671960555455836081749131460608	8
14772066646249520296394239624609792	9
7854692012922976629024898867527680	10
1733682568283424050289211617574912	11
143698285034150052313250618408960	12
3671364086557678901281907802112	13
19728979597377021147304951808	14
10072071724722072768741376	15
73786976144514351104	16
17179869184	17

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n = 38

1	0
84428232354561996	1
1421084711666109180684240	2
31710286852534775338902034944	3
27590510890252567973883416062464	4
3359226587091202365713022396745728	5
103105527898665249997416155693727744	6
1077575892661766160087537691827634176	7
4508199947037195612812437596290875392	8
8213198407224912948771515725452148736	9
6730586739070899453483296762203471872	10
2454879637027050248742033222486982656	11
376671812239455963009975310325121024	12
21602793603019069780917959742455808	13
371656448792038205624261504139264	14
126465905476762564288536674304	15
362606166645254273410007040	16
1180591620081756143616	17
68719476736	18

TABLE II-5

n = 40

1	0
759854091191058040	1
35527129569391142978504784	2
1554076193521951078715984426304	3
2242172681710993093854403794762240	4
413633110157770969153195026080292864	5
18355333798828085859088261915740389376	6
271771274272422005673581640752765140992	7
1607573040942217275141404912122015973376	8
4196314548141550483898166722837182152704	9
5072500387619619140133334801808161243136	10
2865401789693082889199622747870015258624	11
735653682900469128576605472027372945408	12
79916193330140683047810724293481857024	13
3211031448946775664761876531323600896	14
37478102283360831622656139135549440	15
81015414949189133932894057660416	16
13054019158029729761146699776	17
18889465928798521262080	18
274877906944	19

n = 42

1	0
6838686820719522440	1
888178351313257025143680880	2
76156963253699305983254596098240	3
182000631462621471757727169621507840	4
50676682260926711267065429218103861248	5
3226406811308513441348904983961972326400	6
66864824443324079866102620631010504540160	7
549655459584178887026205670472905596272640	8
2007175890517934827347043797591394313830400	9
3461746543872273295629705772362853308694528	10
2888509627201748899454363280169778649497600	11
1156900013936353413004719503992852567818240	12
213365886480869009476348096152757610741760	13
16650134150475140633667970379806251417600	14
473313290826732659527682707174979534848	15
3769282849306521057374563454932746240	16
5187949825037714354992289824112640	17
469948033124539767970468986880	18
302231454892387299491840	19
1099511627776	20

TABLE II-6

n = 44

1	0
61548181386475702044	1
22204459846247226250477761840	2
3731882601868539148753052935413760	3
14762118799251366388049087559124243200	4
6186409283287913495828286189212018039808	5
561700496824122745104735474437399335944192	6
16137271790726031164409154841185453251624960	7
181818713540547860220809520735474813170810880	8
911314581749394580309238395408887166812815360	9
2185040342172233710165234767254828423738032128	10
2598198976577956008906582718608285482103078912	11
1542748630930845649763595926201613357091389440	12
447942780207967442232026959822357515070341120	13
60319581105778980133848290550977453610762240	14
3420925865312361702236352709165217111605248	15
69338256094993148207373392986014096556032	16
3783966180006570530209562133484535808000	17
332142046384479973470922576274391040	18
16918185709765496963166539612160	19
4835703278411237698830336	20
4398046511104	21

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n = 46

1	0
553933632478281318484	1
555111506219308312950721330656	2
182867299006198972411243623305612800	3
1196774692027741337458276064976132954880	4
753273205458785098408026910884229593781248	5
97081192922862770832513786109739685582385152	6
3836496456821784336545689394570223630103216128	7
58584443107770934537026416765266177701364039680	8
396893404850221215363669647377163452645049630720	9
1295889048385403236487838306304442118929716871168	10
2135406861791294847364452836494584827005561208832	11
1808808490184014996264447407162420023476185202688	12
782854981070912529160409120502384573448265400320	13
167748474242489578267042445929216676933696225280	14
16711980843193286984925085854360626931966148608	15
695350559704315021906853317286928676187799552	16
10111569728034439612939221453176437701869568	17
37939642323605799990146648427684199137280	18
21261405107846214930806302772612300800	19
609055638185103739627360607010816	20
77371252455138355088195584	21
17592186044416	22

TABLE II-7

n = 48

1	0
4985402692304531866448	1
13877787750482325793793279386912	2
8960630600509742095126393619994481024	3
96992787565420470409268222155766171022080	4
91552427666396226323941500863262517526099968	5
16686930436954409253274600219009763354321793024	6
901417261406938813128944246221776947974544080896	7
18485145262286424220302767300566978094165665185792	8
167155353069083467241305514678438222343914431447040	9
730879726448587317116505055259223663864040388558848	10
1632287596022932727006300352610124658173028161552384	11
1914297376556192079044598182741777579348647819083776	12
1185218137633402420536387717355337617702902638313472	13
381179143993492935228604617760913225631377873960960	14
61159554161926705776754609181598161054566006652928	15
4556770909047899393086780219557232488270281048064	16
140172250689010538205600238840727050174737678336	17
1469617520097726258834794837106661584642506752	18
3800746851345946072400789846481207579115520	19
1360893762902099187033274710414397014016	20
21926018990512992848739103154372608	21
1237940039284553442155036672	22
70368744177664	23

n = 50

1	0
44868624230740786798128	1
346944694656937928113495454700768	2
439074389688599103381208754135111431808	3
7859207035411502926933862876578364093112576	4
11112716177133122635886137586044237213431072768	5
2856333555970319131931643060311011660193918930944	6
209846236077380510238102845038841144246819761373184	7
5735394349126752048413545122760857937981576056274944	8
68511180556682536979177585823499102882064166381682688	9
395685301961571212094489594681794774651645222587465728	10
1176270283440967429483865894836884460087001604679532544	11
1865049203648341867422570498887991598257490278045712384	12
1600044624113181164403368099014082930123488014687535104	13
739230539402039953190792361385492444141442499889594368	14
179469548944650017027516393823179932736901691681538048	15
21821377973386222967465690562465846449264812395331584	16
1226790611350600556041894789530964327773586518441984	17
28076050587784453871272184363763524969959002210304	18
213065906761901989517908261022517959865252446208	19
380533315516953963808053037989608365014122496	20
87103405889693589815667735238088629157888	21
789336952291456267339929726870355968	22
19807040628562636329921282048	23
281474976710656	24

TABLE III-1

TABLE III: Number NR of Permutations of n natural Numbers with j_2 Runs Up (Tables up to n=15 are given in Ref. 3, p.260, Table 7.2.2).

n = 16

NR	j_2
1	0
10761672	1
9453340172	2
450403628440	3
3947368484790	4
9228238224824	5
6230311951468	6
1037611984488	7
19391512145	8

n = 17

NR	j_2
1	0
32285032	1
47417364268	2
3266265481144	3
40030352647510	4
133089568351384	5
136363484718028	6
40485427573192	7
2404879675441	8

n = 18

NR	j_2
1	0
96855113	1
237571096820	2
23480284103492	3
396202094120174	4
1824258425692814	5
2704352279794052	6
1289098837188020	7
162339237202073	8
2404879675441	9

n = 19

NR	j_2
1	0
290565357	1
1189405165908	2
167687984079924	3
3847582256323470	4
24028863623822694	5
49750647042865188	6
35562596236584612	7
7916162381187321	8
370371188237525	9

TABLE III-2

n = 20

1	0
871696090	1
5951965440609	2
1191656966048088	3
36808184099950242	4
306640904681607804	5
863018184171651690	6
881693472848825496	7
312387741663107517	8
30785539720074938	9
370371188237525	10

n = 21

1	0
2615088290	1
29775517732665	2
8436830209386360	3
347956854424225410	4
3814748160697088748	5
14285645441047550010	6
20129547566105595960	7
10600752445365780765	8
1834476221333853890	9
69348874393137901	10

n = 22

1	0
7845264891	1
148927275340835	2
59563995267159825	3
3258164142958824090	4
46485668875182906558	5
227675620501286126358	6
430514022461011889490	7
321119624533957444725	8
87858810432172127735	9
6959755026257457591	10
69348874393137901	11

n = 23

1	0
23535794695	1
744793282001995	2
419628657826253805	3
30276501210903974010	4
556956655628435509398	5
3517611093018914521350	6
8734466541928039605930	7
8903145796765371676245	8
3596035145415015095315	9
497590097280095120351	10
15514534163557086905	11

TABLE III-3

n = 24

1	0
70607384108	1
3724460661698570	2
2951551677141814540	3
279622198081182080775	4
6580670730076350213528	5
52969380732415550399724	6
169710720152128653823800	7
229963677422363674949535	8
130646688340242888544700	9
28429567769957073003946	10
1849604577602098359868	11
15514534163557086905	12

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n = 25

1	0
211822152348	1
18623856670943226	2
20735350953226673180	3
2569727712919191388695	4
76861333200138765641208	5
780731339742471058185804	6
3181293371070916412153688	7
5606489717701469012380095	8
4321996497843524281945580	9
1380901053210555864351066	10
156259176364676554292748	11
4087072509293123892361	12

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n = 26

1	0
635466457069	1
93124155264220134	2
145538557662676520006	3
23521521084384029288675	4
889160036321152675661103	5
11302427414654205241033572	6
57868907982715869938720772	7
130304552282705053744152183	8
132576340918340182468386875	9
59252897602326343124991446	10
10498466322440340070488534	11
570954341826357760187269	12
4087072509293123892361	13

TABLE III-4

n = 27

1	0
1906399371233	1
465636027516070326	2
1020818635054548482990	3
214604460912709793998195	4
10204147779051591959468283	5
161158116971643110944014084	6
1026267603545896922455281588	7
2909604284598576352915236375	8
3821996000275514004340872455	9
2304921576995574665371915366	10
596982111030085880371184958	11
56267723835420304286635861	12
1252259641403629865468285	13

n = 28

1	0
5719198113726	1
23282227797564632455	2
7156440074014708998428	3
1952877339550533664126545	4
116323110326908997640116818	5
2268526032875237505583143903	6
17811385807763100500989435080	7
62804751684272457991477679019	8
104623571135819105964544176770	9
82801317119386694011878074781	10
29865039592660997906140661596	11
4391603651035937009021821315	12
202614181824158919227551278	13
1252259641403629865468285	14

n = 29

1	0
17157594341206	1
11641287686974119151	2
50150957985244514167916	3
17733337737583126575104321	4
1318611760387009647323815898	5
31584654413262449530102973463	6
303467203642450307604171828648	7
1317040179941315192868972634443	8
2741504871791832509224071506858	9
2785063370865311633894881338101	10
1349306447586296503936259814956	11
288980328831864412662389502451	12
23036997513396038855231169766	13
441543893249023104553682821	14

TABLE III-5

n = 30

1	0
51472783023647	1
58206901689917808317	2
351347738088885952154187	3
160753511671908763001800957	4
14877129456746351778639165619	5
435654130819765027190491157081	6
5088947179662216256074327978591	7
26941691113639112892835112215251	8
69210114903281915182554002878061	9
88642884377881701913257294675559	10
56099618632272624295587907786897	11
16669653353900685844113556265967	12
2066900577021015112403189095937	13
81915765444409786597750311107	14
441543893249023104553682821	15

n = 31

1	0
154418349070971	1
291035949687513703701	2
2460947546066139528095551	3
1455213950761312129867909101	4
167185001280991862351070442863	5
5961046289791872389049168354433	6
84175982049689014330543760506323	7
539431903806460379275386155316723	8
1692175858753303968968217625696673	9
2692021950774898722369051222723471	10
2176720072321087377931094825854221	11
865538282905698140467542168944351	12
155824235702971523099567443186101	13
10643159505971944260947515405851	14
177519391579539289436664789665	15

n = 32

1	0
463255047212944	1
1455184226569691576664	2
17234490793104539566668784	3
13158449245503462657013570684	4
1872504934958420664848736780816	5
81004486794195170167011667907752	6
1375899610251380790350090606329072	7
10601334059554539691300808568991782	8
40242819873409681099526927217987632	9
78530747130065824766336904811249640	10
79676803121908895638474744444605264	11
41228937723532239913068407656296764	12
10266022344320118106961116148635184	13
1087772804188043999065315162700184	14
37432579656881550755379154697168	15
177519391579539289436664789665	16

TABLE III-6

n = 33

1	0
13897651416388 64	1
727593503049987 4271640	2
12068218071007572833 0828080	3
118874139970151881941855524540	4
2091335706643471041710443 0285392	5
1094253436893622466797823891978728	6
22258583889654615258591592453091120	7
204988871996952028978415376586783590	8
934234922547656575951824554245633520	9
2212545167959117855486451982088069288	10
2774935437364694496880961979960916752	11
1827491474307394954211457635853471740	12
607014105084901108192497397263524080	13
93141545387373984614660836610116440	14
5511501186115504069678014446412944	15
80723299235887898062168247453281	16

n = 34

1	0
41692954249166 25	1
363797182352187621629 84	2
844986267086414594669674120	3
1073125678610538982141632079020	4
23301878123003561163669512125281 2	5
147063018921450904079651124922874 80	6
3568580805195853006816281885279200 88	7
39077239178516221825463016585840523 10	8
212352743523536594357177279326423579 10	9
6047697236535632360449285993753395784 8	10
9258660224285650554958600130624598604 0	11
762115766686965133209770226759068777 72	12
3283680410605888450910054844879639562 0	13
695020355182815331117264604253804532 0	14
6365642637064505492333226308893834 64	15
191983724331308128450855955051971 05	16
80723299235887898062168247453281	17

n = 35

1	0
12507886274749909	1
181898724593547408146920	2
5915995261151958725552608360	3
9681790722973270447925439586540	4
2591107861174265741539328767835452	5
196774375347407029982827145309804728	6
5676409849420971499199655302749125880	7
73568468213869283116919691966487291030	8
473809243216048728564470260574717741870	9
1609780809310141346665833705610490841368	10
2976169464700688158103378669169133088792	11
3016328643631690899619457582572623776780	12
1648709477550555014955485034876571464860	13
464250335851487522096811122823974480840	14
614347134858688868993268877812799159304	15
3179803345119119020821115175229038321	16
41222060339517702122347079671259045	17

TABLE III-7

n = 36

1	0
37523658824249762	1
909494035727984107481597	2
41417605688526111048520813040	3
87307680369332840834369981921300	4
28763594822437201459026603315026552	5
26222844576045648033315236108223347764	6
89671958374304934177599853883362396944	7
1369868566473618214470827524788015590990	8
10400176517168442221946409098282895625100	9
41860154130185796665578502247590509280518	10
92598609827767947836365201175047423662736	11
114098419131901218545830354463514324573796	12
77694770973813585299612129349966291095800	13
28301645037648133275406887875784403128100	14
51542284690223461483690130719645953043 04	15
412107077818275362153441189846554061113	16
10982182147240476636745493314181181538	17
41222060339517702122347079671259045	18

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n = 37

1	0
112570976472749322	1
4547471454444320561899893	2
289952343628826072831085102384	3
787011651494651350840785461682900	4
318844158097150535592654995959088472	5
34844832953976791671032761093094211284	6
1408027645439669565463563474847796300496	7
25260548714286218197911264706830237779598	8
225000725155672766506398323363135328696700	9
1066266414042933689972183910968492816142678	10
2799530492121635546885655662987538892731216	11
4148841015886281733354871678038522045623204	12
3466939845875781425639491746011261754472152	13
1597695415829931717982921041897410601672900	14
386194242840877858802694508237177679458224	15
44524904382137175841277637696723855842553	16
2032804686426518130899857025382557598282	17
23489580527043108252017828576198947741	18

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n = 38

1	0
337712929418248003	1
22737361212205779355725735	2
2029816471959779172396138413157	3
7092093386104355765824832793320004	4
3530109076962000780693587733938777292	5
461591620670321356184427579101120135436	6
21991535505444463273729271150044299789540	7
461813963987978109370153459937613357164574	8
4805485300957793145777704702743006238608858	9
26666608472859390053037430274237920384233538	10
82515730357527490307897206735177772407243494	11
145713982778981576537156626895776134531548340	12
147542309045167761025879608956804853963849756	13
84469505363701615503539119421148786747707772	14
26351280270536599084729819132429203478261044	15
4172681544496671814381023601652130999011817	16
293772685935614013787883184372008795152635	17
6967528538780149398024230733467033861263	18

TABLE III-8

n = 39

1	0
101313878825474404 7	1
113686818218694355835556783	2
142094883739996692032710635670 89	3
638937946020419148259401715691 01060	4
390439626481651392606042100571261503 32	5
609953412286911365225697898486484752 4844	6
341874414719095304366734184307293620 364436	7
43786342399262949778621113265404902 66746718	8
101464127925933588175919765470744612 391188930	9
656108483949203054029340129813856452 841081458	10
237786075073460129803630949984537133 2282804046	11
496310125519497925835527098015724772 1804604404	12
602363810312527161921894221837459694 0465620172	13
422012336408935998191318977069497306 3249722460	14
166158474202365072666201558731679317 5303170084	15
348508733132682962552412331913953950 793478313	16
353181332747265213688620530629330938 24413127	17
143288929967732158287842927462631543 3477271	18
148511507181149800178771567814058266 84425	19

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n = 40

1	0
3039416364764232180	1
568434128579606944603313654	2
9947039765663533872535168945 7028	3
57551306453471922251716948921 9623477	4
43146429676247983122625045594702 9786512	5
8 4262185140952665178982488948996 76182600	6
5292803642103895634111951197200755 188637328	7
1509836424467243972328242471588706 75043805106	8
2120527018111042959833304104454578 91156776 4184	9
1590902484937786948631045780097662 2369877678148	10
6715685846193068788139258096290681 32464850407 60	11
1645011641423627035254990360013025 0569392277 8882	12
2370847476123070225942405045984728 3321964081 0704	13
2006908728992199705253287521890239 7906029501 3576	14
9793048400771613232756757068446529 2130145219664	15
2645505087159139430418774723901161 8953913315085	16
3675695796544208985777058180600335 939408807636	17
2296075704616935054108121484758391 40160724662	18
4877862777038448969332496938353773 541124388	19
1485115071811498001787715678140582 6684425	20

TABLE III-9

n = 41

1	0
9118249094292696580	1
284217075839585658405739 1110	2
696313247225076236927467545490740	3
5182999574332798604171187360418150245	4
476452368245238915860930443907235560 2896	5
1058484769586112859669464749312106683969160	6
8164398874995310197418041892706851876267 2720	7
27043348162890160394449229328280211106492 57330	8
439136207628312017704205599164498955208388420 40	9
38074111623537820432885230411850980582192205 3156	10
1862788241611963210998238518166389152066709500440	11
5321352555873819970002542357364885280784800205730	12
9033306811810092866452478200179606588033066351120	13
9139221780649677461553900877860315057823526743560	14
5444135479029839748458539717486711804758041972496	15
1852321518839677335313871365732036346780591594445	16
3402897598517784689356990142331047095106147879 40	17
305496548886347913614862398577208063822399658310	18
1108666930151273531447215974499153728746749780	19
10364622733519612119397957304745185976310201	20

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n = 42

1	0
2735474728287808 9781	1
14210854147590997597702122170	2
487429789089359430518588294 1906250	3
46671367132648065105833147607855528105	4
525807994929292630986399980126897105899 41	5
13908002238775491239619930178668629915288856	6
125535588956729380254312076163607887527519 6440	7
48178079573162006423866561169107208887629538050	8
9019671649010182346241137117332485426621694 32010	9
90055767184880599316255712645670535192396564831 96	10
5083969299801809614386538430431565641979468162639 6	11
168426790487472800259030090779283525908887485651610	12
334362277368727446884260131480052427650234394977650	13
400537033815992039381850298460643235497378270830040	14
287578082998370839202415442654272161699205148813656	15
121012100391037589298401691961511029296097984314141	16
2858103526436934243057430778974699195389684192790 5	17
351236554975732208629980184698843132800309087305 0	18
1959862847076392358007534158915073145331215329 70	19
375095 3225281246912369641729920138112689675 81	20
10364622733519612119397957304745185976310201	21

TABLE III-10

n = 43

1	0
82064241848634269385	1
71054271832144879303634202090	2
34120625248712768594209893273986210	3
420217778917904755347485020256608377945	4
579975620904731928298638305158253904444921	5
182486614687855122534215572259098259637633240	6
19247578410672671775335409329901242026586612280	7
854177317651638335676938921200632759597407647170	8
18390006202021558624878691113328509741659587197490	9
210764323045873696195115725637506088927924852515356	10
1367435626761153529804666406819735275078550120037420	11
5227463622147181929353059955568401276118080773818170	12
12059463717730151470537565183988519012916303406125530	13
16965370418563408292221820759039492671827720173713560	14
14522439046373384566620782900731442309638655404843896	15
7444336308884690517276241146581129907161695268130525	16
2210457238163302878054117692256255011347369310618085	17
358605807455975656637687130656547894767289097726090	18
28717658402141862714028194301699373234810285024130	19
937734102054210055543[illegible]29194658701824394513802701	20
7947579422597592703608040510088070619519273805	21

n = 44

1	0
246192725545902808198	1
355271362525358312312176055235	2
238847147857590833809762094651784980	3
378322473395345170565350948360612891275	4
639443945221417787772218333244977424212206	5
2391465186431972746578657503438499754135914513	6
294370761215413585428591822688550676447565814640	7
15079194173987359187992688530977892931926941758010	8
372472905415003848935972482025658769600662163225900	9
4885800939013886585719397516220840611027911582759726	10
36298598845561626197994989046516551372148924368713848	11
159402738715663772359724493432424472679601571866240070	12
424927329199510546362222399123489637594983726667934540	13
697006625339751415473571410139950110702581042941827250	14
704676166716026045948571581308267101676214120155864176	15
434454805796048786436186135546686036961638464111277973	16
159253702733447196421932771841361354375936573821068750	17
33162530019340825198081483064598567208516020411428095	18
3630229329875362242310909492362110819528624800023700	19
182035390194931925847433868489503640974226491031391	20
3154948221334326652886333325909892509822870181718	21
7947579422597592703608040510088070619519273805	22

TABLE III-11

n = 45

1	0
738578176637708424638	1
1776356822966886034488798220491	2
1671944245857636851000827149604704260	3
34058078452176694986772929494842283850715	4
70475029983398189081084369291088498727430166	5
31306458364990927753365101778003789127999043673	6
4492088304197026909319394380438292138845836484016	7
265177423794247513758733459707280700136184984325370	8
7499202639756719187047272437354897624506535470516380	9
112286115260081718372442632373304780841203359481827646	10
952126995984250680611150288459180856224095138466651928	11
4783637642493950085349002094833975947177315632767706406	12
14661092662700060198974494644982709668656592057359033980	13
27861884060443980883253574078281366687084557325335811970	14
32997067173632830071982861582795481923203934411901025456	15
24202474925293974595674144611356378643201066997854271573	16
10787337265223237312001880641007879846697441653072741926	17
2819550638050082496548342591803760530474458493433527015	18
406879184019865729034777334718910859629744530492349060	19
29244826977244382413610245562242314197115034932429231	20
863804334297103749463847806972139941819289381939438	21
6667537516685544977435028474773748197524107684661	22

n = 46

1	0
2215734529913125273959	1
8881784146593291767865453361889	2
11703682551633199599333204087959969951	3
306587911895178702718145397712415138122575	4
776485478720110617606438660593282650504208281	5
409450584794300997411584276039237356119447623559	6
68414437689000104255651964065248507123911515701449	7
4647270941932315568087370040817358958619365664535786	8
150174995445410842452901446641254195169573538485246950	9
2560486891735147503871571635647982633520947006822322826	10
24706073789139808613367522443893779214184272171778685494	11
141489861969986517787781509005410558372587079003925654494	12
496305892385274577164640399406046655944451613836815751986	13
1086555398344176589394869046524831117629927411524560192750	14
1496561111410165407246779468397443173299759441299640592626	15
1293637680139193612736989695916692724073694227107706343749	16
692188978311634975663829702382908716996024328829651497859	17
222984083525308662804309362947825817941226822440840660741	18
41244243919225505901291399380271368299830163130103356475	19
4047192194206079782201461411084310899489928145676041891	20
183367721260997373294996683511013588483804618085541989	21
2891452191142160772376119702261238494346452991628059	22
6667537516685544977435028474773748197524107684661	23

TABLE III-12

n = 47

1	0
6647203589739375821923	1
44408920830458778155504778863641	2
81926150896366554113586678964760988995	3
2759759354358673652447281907575254641901215	4
855299 606573233584374114791639180930794948941	5
5350811079559836948584427380291443804970970604383	6
1040137885218007797746773326364061676966733954722741	7
81192868018897367993666153543983054524494384799554730	8
2992743041720775473647748587408350476980478201155765630	9
57975124598909601169984244854562752768813946220855693946	10
634812356335329433208113878736404470397938882128290159838	11
4130192320189018351415358263788714660455099507220829814206	12
16513036057742116974776483982082291994697109311680389702490	13
41436224399686612635744010337341035530156927210948560629470	14
65951391623910306233757806357767698489631236087730941840906	15
66635021227156035736269131459609950667228060555348558825733	16
42337541762855934726551895326235503231892570688545691237551	17
16556678830176040231725402857664459867777684376266922421725	18
3838366348102881358193458205308841543105644586482437509935	19
495888831316253318280591112896627693277728359013544569331	20
32167965179459365744893625857479449701743167451734556873	21
863586233645386727936912120646710086180808856965430011	22
6096278645568542158691685742876843153976539044435185	23

n = 48

1	0
19941610769218127465816	1
222 44604451418052315795806304740	2
573484965858161588522567439458818059528	3
24841193161414813900744194222014846977659730	4
94190527287125557700560706702426425169778585736	5
69877004686721089974219398191079419159062030967796	6
15789346666054711259402054853771125687674993291994520	7
1414603306533449513217968130017725962256306762098280863	8
59379096701280552407110873920622133752888411750745743600	9
1304267164787004113305453850980659971977526738471486776136	10
16166012559883136195376193822010497143910570836913777412816	11
119125116913108694615586803563127978271325959733727999351100	12
540846396923384580401518307583362321047289240081449607693968	13
1548424264803496222906882463406618162263190184662796442006920	14
2831781403935265133325628193500278328251549935727681849028016	15
3320129358102624385270764046249179246342257011817928452544591	16
2481339280107298251473353308312391873124660882429327575700280	17
1162985159633640640019014544974646557122377740872970115691988	18
331819754707948815518524301441353878726674667011751209526440	19
54876739216892318273245359476541309312337663997897264931986	20
4854444321930525954994563702148010190119054713519397930856	21
199701206411339231481629174716499202386852235822117161860	22
2877283797277881665269245591855341886779323905984745528	23
6096278645568542158691685742876843153976539044435185	24

TABLE III-13

n = 49

1	0
59824832307654382397496	1
1110223023174404356963012894951236	2
4014404530969726982052273971227203825256	3
223594824821299367893415695830590893157437746	4
1037089447884837727262197541495571270746670832296	5
911980300964284940857473483338724653224257988839316	6
239355772159542628130102721141445744404851132494758456	7
24585093997714501907525128075329559631738164727598588351	8
1173470143133400880158081584652387772097081639651314116016	9
29170983361565502951627857088212523424114713860423594606856	10
408337769491348247666205165733699913525313877926218510226576	11
3398444249384678906469451128450472382524823835102958196510716	12
17461855522845992341615077590265854146788632515808611392163536	13
56802924411615851233132994205625897768672878637012988187467976	14
118753708818063143591232123266640991421061851700814066160006896	15
160536334088221377113796521009227925037822380233089912216475791	16
139968944533397428965899590530920583500839243074113320390223256	17
77769200827946879201330484480435408837273228766310480340407476	18
26896792349213691485450622295908559955808844903933938559835016	19
5568143854972073204388302752951732469072590894031299957475826	20
647755019578151162250729115004694912673820664664512230482696	21
38113220220093421146640695075130525248122678893111659868836	22
934037164117417364194171241683197878226037166869751687256	23
6053285248188621896314383785111649088103498225146815121	24

n = 50

1	0
179474496922963147192537	1
5551115118683788903274820447438492	2
28100881676824131722561981134170699582412	3
2012526042786526009300969510256080808181425722	4
11417151314550888273967802999980338204832834102842	5
11896190401003212902510380987521747771474474017370652	6
3624079853528817964763267336005218978242064533008431532	7
426324049986730524412480772520553114793718590006492547927	8
23107240821459195285951879334881243137691910589385721619887	9
648968225030010989269085527976687012841418521898086224340472	10
10237727215786409281919926667433261190381545894255309978810072	11
95986226010883375348723817686071707228304070581581854688885452	12
556431205351458765885343373148439871526413673804406462501183372	13
2048907484962317509618003616539265680667652028336974719456332632	14
4874226386003890327223988699584014587195047854102508802897041272	15
7554019492454505172988695535370700363248313730007434360183832127	16
7628030738168673424741026525739095148172353971556494721337902407	17
4976994598634925964937721783739918879491698110465187578448425452	18
2059974512382643397549870567786094153161096925215459848258862812	19
524158613895205607719877258126015190745873520598556622414694042	20
77966760536609158816276076721767473466627606626855725528038362	21
6249380046951262087353935083413738024882265202841610307476492	22
234465847813985721850329523734762926517237141308436628645212	23
3098722469513494565501918530520064439995182913641449002697	24
6053285248188621896314383785111649088103498225146815121	25

TABLE IV-1

TABLE IV: Number NP of Permutations of n natural Numbers with j_1-1 Peaks (Tables up to n=15 are given in Ref. 3, p.261, Table 7.3)

	NP	j_1-1
n = 16		
	32768	0
	536608768	1
	84134068224	2
	1594922762240	3
	7048869314560	4
	8885192097792	5
	3099269660672	6
	209865342976	7
n = 17		
	65536	0
	2146926592	1
	511780323328	2
	13684856848384	3
	84842998005760	4
	155964390375424	5
	87815735738368	6
	12655654469632	7
	209865342976	8
n = 18		
	131072	0
	8588754944	1
	3100738912256	2
	115620218667008	3
	985278548541440	4
	2550316668551168	5
	2165206642589696	6
	553753414467584	7
	29088885112832	8
n = 19		
	262144	0
	34357248000	1
	18733264797696	2
	965271355195392	3
	11124607890751488	4
	39471306959486976	5
	48165109676113920	6
	19686087844429824	7
	2184860175433728	8
	29088885112832	9

TABLE IV-2

n = 20

524288	0
137433710592	1
1129493047544176	2
798443654487308880	3
1228293351698595584	4
5849017624213585924	5
99008199114149068 8	6
6039680635675607048	7
1180718345355264000	8
4951498053124096	9

n = 21

1048576	0
549744803840	1
6800322016051200	2
655697319611596800	3
13320910268320972800	4
83699438359247585280	5
191252637377730969600	6
165940629550714060800	7
51451333394772787200	8
4532454646690611200	9
4951498053124096	10

n = 22

2097152	0
2199000186880	1
40900886160998400	2
5354383709149593600	3
142388865157772083200	4
1164244183530822696960	5
3514531306880709427200	6
4185071171833272729600	7
1921767778410194534400	8
296454426512903372800	9
10154238865068523520	10

n = 23

4194304	0
8796044787712	1
245823127001497600	2
43530384737933721600	3
1504204407214964736000	4
15821985449420909445120	5
62010124315168981647360	6
98591920511258748518400	7
63887318214216410726400	8
15537927422309040128000	9
1112756534570217635840	10
10154238865068523520	11

TABLE IV-3

n = 24

8388608	0
35184271425536	1
1476697970966528000	2
35266789418949672960	3
1573853022795658690560	4
21092268709406041964544	5
105800556580541665640448	6
219757197133182979276800	7
193870709194596538122240	8
69408245773147926691840	9
8663235344978094850048	10
246921480190207983616	11

n = 25

16777216	0
140737278640128	1
886757652279853056	2
284940041496433786880	3
16338065648078731345920	4
276715019854807383932928	5
1755407285349861864505344	6
4679921276516885990473728	7
5467487539701384499691520	8
2745259879825134300692480	9
537632406455257720160256	10
31915821559499276156928	11
246921480190207983616	12

n = 26

33554432	0
562949517213696	1
5323642133809201152	2
2297255485017067356160	3
168509577227723121623040	4
3581989288626948308729856	5
28449712272865369478135808	6
95943627848468518221643776	7
145213988479793780899184640	8
98645097914113762011381760	9
28299472220966475647680512	10
2916509343249013508407296	11
70251601603943959887872	12

TABLE IV-4

n = 27

67108864	0
2251798907715584	1
31954800641751121920	2
18489840364946532073472	3
1728743626492555495997440	4
45848534276394672772349952	5
452025811149519397324849152	6
1904944305122746094762065920	7
3669231698969441756623405056	8
3279827854600419268320296960	9
1313104074260058798328643584	10
211493585342808702440177664	11
10576069671449583482306560	12
70251601603943959887872	13

n = 28

134217728	0
9007197375692800	1
191782847024291905536	2
148621728533690781270016	3
17657233072224485601443840	4
581299796593602072196153344	5
7061937904515586326905487360	6
36807470238057209078740942848	7
88905502242922904756366082048	8
102288874081702802932639989760	9
55126912470524647709792534528	10
12954470493787761648536125440	11
1120952152828923980300681216	12
23119184187809597841473536	13

n = 29

268435456	0
36028793126649856	1
1150922262080143753216	2
1193384833751084963987456	3
179693387021452362421108736	4
7311084987495490092781273088	5
108749227205309443804011429888	6
694848592376649140163437395968	7
2078796153467356003638221733888	8
3023738006306208010972826697728	9
2133391941086867476009195667456	10
696795679144578813533414752256	11
93917108442490831730498338816	12
4010193615745440680463302656	13
23119184187809597841473536	14

TABLE IV-5

n = 30

536870912	0
144115180022792192	1
6906470321102155415552	2
9574700804298603161976832	3
1823188336557047493418811392	4
91326887590374928361797451776	5
1654088710649251034926222934016	6
12857565113311337343479181213696	7
47146211055685496027776114753536	8
85420313967732432263115194761216	9
77172002766973164581930571661312	10
33790231828164831332875519393792	11
6622618894372234506193445322752	12
487953855010835665974965829632	13
8713962757125169296170811392	14

n = 31

1073741824	0
576460735660425216	1
41442713036473547882496	2
76770268192416379181203456	3
18452101484069342806913581056	4
1134209606152197137703364460544	5
24892452813306638127841272659968	6
233840549894018665089412689297408	7
1041495275702008988652157783769088	8
2321307023078560093623393387020288	9
2637407514518466375696739718922240	10
1505513588778714433226387610402816	11
408719714050831916491158214148096	12
46775802412164571178266269843456	13
1725280447746262076810021830656	14
8713962757125169296170811392	15

n = 32

2147483648	0
2305842974853955584	1
2486724191119439779201024	2
615239656078279345694572544	3
186363501277311421169484693504	4
14016461506475891194192472309760	5
3711178531509336876543845106450432	6
4189512948943818127731745936637952	7
22488363760940460437169443136602112	8
61007074321399327713598076713172992	9
85878649596348981388808994460532736	10
62506401275873810154400699838889984	11
22670821275551345294581214451073024	12
3762040751845599491938404840505344	13
238861623081046147017365734293504	14
3729407703720529571097509625856	15

TABLE IV-6

n = 33

4294967296	0
9223371965987815424	1
1492101384162909439918080	2
4928631403942459639595008000	3
1879016004175071195337211248640	4
172483898607088857017207815667712	5
5490845132766709986691873408811008	6
74084599281778490698040992008765440	7
476012267828973196040489657381683200	8
1557466942842093460829503181312491520	9
2682422257297868850830572875402969088	10
2444818776180810238982515735199219712	11
1151998964647199269048717874277908480	12
264032889980536202836343836691660800	13
25976052451659381870212996231331840	14
835925915762195387327217510907904	15
3729407703720529571097509625856	16

n = 34

8589934592	0
36893488001390215168	1
8952885006136436273971200	2
39470830070296238581077770240	3
18918304458253215904001582694400	4
2114903167385267992894586857979904	5
80666477628089894668064799668043776	6
1295170491163790050902493340316467200	7
9901743607993530361293551689028075520	8
38765535142105440353237898144356761600	9
80817826860342423169885647797240201216	10
90864717715913871945547252149616902144	11
54400160842635283385091822083217817600	12
16608912636632607831807370421589770240	13
2363478913432998673124452907089920000	14
130653839111027779875322945274380288	15
1798651693450888780071750349094912	16

n = 35

17179869184	0
147573952289028702208	1
53718453734946660740497408	2
316026274227547865300567326720	3
190248756994430157481704926740480	4
25851795620079546312335081863118848	5
1177973459643119689189482693086150656	6
22416723888810528602469254238092394496	7
202839624275995557470431303868518236160	8
943640344177998823206748341600612515840	9
2359475218059114915036052723704635850752	10
3237384974366384427901647472954928267264	11
2413916076783569959413407147809449181184	12
954651001409430569756432770553473925120	13
187166755859418215016385180163825991680	14
15998317418717882321632598784229769216	15
453115674910413558148408347692367872	16
1798651693450888780071750349094912	17

TABLE IV-7

n = 36

34359738368	0
590295809740230361088	1
322315444776153213361455104	2
2529821747432431322226753536000	3
1911336305622672915045465152552960	4
315168015122809739842545310452678656	5
17112071529885584760148799667920961536	6
384582998333117090801676687057373626368	7
4099447714744130606517148554395176140800	8
22523920120527896498602730301645578567680	9
670067003041485093021011333871117889699 84	10
110585892437620835080144277482783180324864	11
101538437688769432079568355518504817917952	12
50869388807299755547314189053591761715200	13
13252210687057991008543017569342571151360	14
1634946692555481524390554242078308564992	15
79399202621825590263576278958459584512	16
970982810785059112379399707952152576	17

n = 37

68719476736	0
2361183240163512287232	1
1933912148418640707770646528	2
20248565758247511327428233396224	3
191196727886902269658799227378073600	4
3833622261725529046816771284551073792	5
247448201796468430138146828112210427904	6
6546905618517241902310249385280160137216	7
81866301830389809824143884407318016688128	8
528367908990696411495880428566419918028800	9
1857054048740241445122471349644434192990208	10
3659161923065127681454979660393473162346496	11
4077615981597076090110652850757306610089984	12
2541265701180856908200049204204122325123072	13
855390819877437530182118228562602989977600	14
145083768971181345840298858731903872139264	15
10874306351919477690914364694979168698368	16
273152989053738898836387226361656246272	17
970982810785059112379399707952152576	18

n = 38

137438953472	0
9444732963127950311424	1
11603553170742009806041645056	2
162050411254729487122074527858688	3
192474735841770121927815120782622720	4
46540695521539612112247633781198946304	5
3563949003955421777151291646969273909248	6
110689246739391112760279514039175612465152	7
1617625356554395898685415405807887823405056	8
12204684216421724426400486259474758694338560	9
50365811434117847199620217406373110770302976	10
117532750933406927476879053443754302984159232	11
157246282444435765883246689365198596135190528	12
120086831412228906510929211925803104424525824	13
51074381608131694987464038898919312950558720	14
11485807166097303308346509307921747828277248	15
1240229029792350316532881552020714968580096	16
53330733013612511121767398928936299659264	17
583203324917310043943191641625494290432	18

TABLE IV-8

n = 39

274877906944	0
37778931857597042524160	1
69621649590105768314510770176	2
1296786207292470383300195597156352	3
1929770921166597833378935518189846528	4
564070113597886678882878243877083414528	5
51151884834457474407148769169662206279680	6
1860126672929143348593254515801041647173632	7
31663109092985121769823906127443019907989504	8
278063816816076802400403448711460818178277376	9
1339936851662605402493254021870228852139098112	10
3677004816781769661838640978558446154714972160	11
5851394607556433825117599725151478044277342208	12
5406632951320074338788224895698068673644199936	13
2853186593778468821244142498162413537186545664	14
827215263788298960754264647943769747059900416	15
122568437175621033920543537924156543729664000	16
8121051537451801983048034121545281630633984	17
182153925387695315035143479168577682014208	18
5832033249173100439431916416254942904 32	19

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n = 40

549755813888	0
151115727440833530560512	1
41773125758218148338059549081 6	2
10376656794425826662524258143436800	3
1933920637029933738605496144100746854 4	4
6826734490809638081595906992070696370176	5
731920350863145468708803359203829223522304	6
31091975772562187912077940251227883718049792	7
6145790038240316322230684186731993578759782 40	8
62578647363672087269441949090329628015413166608	9
35039887072898854902859657455374251110625705984	10
112366978932689369129005955879066827051663097856	11
210968336864975594042475848510873567626650451968	12
233305247142752155037716693231666615481920389120	13
150475193229194956702782973693249230199326769152	14
55002754379010254956577893715824767277782269952	15
10785048974277506839332597472971480463287779328	16
1027768478401991074912990455920569401080807424	17
39406055314539629903527588694587078439075840	18
3876359837720830318280146240021751356456 96	19

TABLE IV-9

n = 41

1099511627776	0
60446290978477459898 3680	1
2506393136775004211124203683840	2
830278749494219896521123859896 72960	3
19373449337720942614041291492880809 9840	4
825203292871949364381185877095195 87966976	5
1044486021231751606628952833162365932404 7360	6
517233461834299934248384734718149528523898880	7
11839721463146624077817180042398285534718853120	8
1392926118152969020800144718101428412619738316 80	9
902292675067486191112874055710792574326613318 0416	10
3362565348769623102250476432749714620341802762240	11
7395415400345084720297473311226848818171184414720	12
[illegible]697071972971694251693205138149768747893527674880	13
7547224009731626716573806222809142907244768133120	14
3415315265649472682341105309532134085081627099136	15
861716454536527527146509357523453241251824926720	16
11249500804241422657219583872394086168192352256 0	17
66362724939624613108990006499971559860889190 40	18
133723605294502210983703351043848240743055360	19
3876359837720830318280146240021751356456 96	20

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n = 42

2199023255552	0
24178516391830788610 45760	1
15038381790240597088179983482880	2
66431322974829981736849955925000 1920	3
19401678815203746090523009704117298790 40	4
9964434552344099388939162657919569147985 92	5
14870365285106107302119695427401681817567 2320	6
85681914752936893978302625487758549174557081 60	7
226563056344331031691167243865841027366560727040	8
3070005551421457019467901757220415678072729108480	9
22914876311421228050592610636197508859618354266112	10
98747421871820678276586245528151665753525929902080	11
252806976686825418568242881881392932438603244503040	12
389844661648728794572169316847823106031757725532160	13
362175727913552521020919058618371049687852431441920	14
199856776617562646433801044578738005609549284769792	15
63451512110736662746392371251118751053378318499840	16
10943551925819132373771125054249496950563846225920	17
9271484030250588892473370570435370975629200588 80	18
31894034187629933682944136641742553574077890560	19
283727921907431909304183316295787837183229952	20

TABLE IV-10

n = 43

4398046511104	0
9671406556822475397660672	1
90230385037657510669155481681920	2
5315062258112637441040259133388881920	3
19424929778244936584130907188691048857600	4
12021347002903091628825721121527070063591424	5
2112740887027121730402468764075786118818168832	6
141403469537379801482898912454360166406385827840	7
4309476184030888184182427478402086575369397207040	8
67064187437037416181637216241054339245618600345600	9
574737406533960528560799174412414755507276563349504	10
2851150527463542067700514716035787664136607757238272	11
8449182409422053770029453593951097892720677025218560	12
15213369129840438363680869863722726820345471471452160	13
16712941762137507549210111511268478081111938826240000	14
11103701314638187459153581188558439825447658721378304	15
4355775954558235644149152112904155597519904961462272	16
965031478326118730173291843213050649700703330631680	17
111836502795686164407796683547400888361337885818880	18
5911503382630491793554450750887387630777715916800	19
107598675283001941239608109209650749883929329664	20
283727921907431909304183316295787837183229952	21

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n = 44

8796093022208	0
38685626227474619544109056	1
541382697082207336913948796518400	2
42523926819532530513727500975414968320	3
194440640023741420789186521215712488325120	4
144916611647297427389769104302740336424255488	5
29963055522472603157756985772949871905489289216	6
2325837739208890475638456662192036246066718310400	7
81529868459602621756804864159959643016027952906240	8
1453330129525551416421487439259541035871976334295040	9
14253763442236029616696875026858428763054930801983488	10
81071835602872141253149935021932028560438670567407616	11
276701753194244239374776087763444298493469757800448000	12
578059619005129242043594520875356113038645387654594560	13
744802158941572240295197263157617971558885708330434560	14
589299626738347104381856159191628767549892222651465728	15
281340798230638261410914146101442568221048673346256896	16
78298892765322630727730027484711379364424369517363200	17
11970038932845024088882608720505638955336466306170880	18
907479152079336658188958131319900835399135951585280	19
28165157892408048706281343590354882018235895513088	20
227681379129930886488600284336316164603920777216	21

TABLE IV-11

n = 45

17592186044416	0
154742504910276710176391168	1
3248297768603919347943094087581696	2
340212528481446450195959651806383964160	3
1945979785529736911520873129693215237079040	4
1745804762168400078404850779875433974182445056	5
424265025498977259312460181263288637778850480128	6
38142258548538898308105773154034025966137660932096	7
1534986926709905015416002798082842625424437983313920	8
31267909038920299115863480117509731078872281414369280	9
349916048967331436977868436572373958684007886001012736	10
2273560613640360071259626566144112547000791502063403008	11
8896754130710665190060326917310124360599425784727207936	12
21443002642834259325341392252015412836457996252537159680	13
32171078291334364323597024749609593068423542840041144320	14
30029620439750690944647356041496390134979836749803421696	15
17226482287440213244935211036940221297664253788241788928	16
5913508920088635581718336596565477907550812709433901056	17
1159551514336014591927109378741616694582605045290762240	18
120089438613088634949736586296335506103320702206607360	19
5720332391877821336608607087394409221761587369476096	20
94513454358941105124342443281862557297280200736768	21
227681379129930886488600284336316164603920777216	22

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n = 46

35184372088832	0
618970019641880896891518976	1
19489793110808722319280391933919232	2
2721830159762315758341594938214574981120	3
19472725931379664080316177763700794961428480	4
21019712418299871469672960791174163438724186112	5
5999067718899407233040207464201805684026109853696	6
623852617592589645227691096264969651867125790277632	7
28774032437234457226731223560112188036624013527613440	8
668337814726283322748917680696514215089329891820175360	9
8511118712293219390525556087647480099098852808795881472	10
62963439902584596197699880065195676136435185313545977856	11
281333940898565216509280284305233709409602483448302272512	12
778339156613572564910765521402634046632812410765584629760	13
1351106396311047597564055803024565223108950217746903203840	14
1475685106733371939406267789321637973414131461434367803392	15
1006115083929476923552860159836916986010302343297468727296	16
419604108572473439881082649919639860243800302998521905152	17
103198046745654910310413522357836213469647118815387975680	18
14079989659211662133406338481786353800793668450590392320	19
960790592137400305835981015448578223933910882757640192	20
27039921559304693971905495853979589408126678310322176	21
199500252157859031027160499643195658166340757225472	22

TABLE IV-12

n = 47

70368744177664	0
2475880078569106884310073344	1
116938785280563178516560917938831360	2
21775440359616069224347850001785890537472	3
194833410690027371117737099839598318038548480	4
252957039879059505207047228071346890678263087104	5
84722637999232196764001458126516375296720884465664	6
10179611116205114762333384386558174017446874269614080	7
537272015015590509083220448066233443867113142995648512	8
14201203235205465714553559097173537754848694228704296960	9
205289732668060476305783011307050445987586668872654127104	10
1723900525469360793507936023755883229751915767745000505344	11
8762841581122141341788384633435576995787673831867416576000	12
27701509144049901364196320569683661203320399653850717356032	13
55321635864989306660226218997387003579291942336953204080640	14
70190732152755711219589517909710023942104360467597124173824	15
56343189454602794491891262274279747125562251593629453778944	16
28185243999692243841906157474986955786910741370813882040320	17
8537170970632094430487622998713814574528393847968484032512	18
1491982007079360678129975240491980073258570807362107473920	19
138913132484252447778955572021344762010779936229953634304	20
5993709509296408063943746894817993053627128259442376704	21
90296776277175597342965870545525768500031709763338240	22
199500252157859031027160499643195658166340757225472	23

n = 48

140737488355328	0
9903520314279664499472465920	1
701632820622102528140068417276215296	2
174208434305910337448280495572840555216896	3
1949205124514658353946344912396054616006983680	4
3042888148154935102587040746650067424224621887488	5
1195223385424896896883474113981797742218509853655040	6
165754347551255730887310199761232341039238498385657856	7
9996643825999192835892636365562063558166336550549323776	8
300142225154577029563567795385457758412987278863955394560	9
4914007809283083518734725903475968868862670153602111111168	10
46711145660634231428140822864124509109723231816569019432960	11
269207493720440333930688465039466199404525498054432843104256	12
968424770818084347716841437886725207600300014608903250640896	13
2213679258830677227090712981315283331445166263185610469539840	14
3241892874457990278910936515063686830574594497028265647013888	15
3038720155900586392237735203880871785342786321664955415265280	16
1803473436353359901195099340939446868086658211660112106356736	17
662635424880326514461403563650968423275007862672568977719296	18
145050989989495371430075239606817348675626770774169139281920	19
17770207620973488231755935948832320590521323780554912432128	20
1097202013314556641487258295500060266424273260795186380800	21
28128489745935709733551417624366157565509971686883065856	22
190169564657928428175235445073924928592047775873499136	23

TABLE IV-13

n = 49

281474976710656	0
39614081257125272659842564096	1
420979736939102931142531297991825817 6	2
1393697644658569449999495398752466731899 2896	3
194991937909531258632988286242790326228337 29536	4
3659067677771529290684839641138425522071973501 3376	5
16845714257430289155164358103371220885755448961007616	6
269390237930996308558788479016908037760546381904845 2096	7
18540948233717691016534869117223781130128892835661453 7216	8
63127404616975155691840276350415791385629020103461369 28256	9
116812296333710571269507435942649590108955374466301150887936	10
1253745706705864809281217348132839378092649657744913466392576	11
8167173478247304467901420662629233912260743744829479406534656	12
33307665938476489416477394956736028399112486864301246409342976	13
86747297952100088114775059635079729302961288202355282349654016	14
145800477900438556239873515127028361875845182905431099625701376	15
158428664166405772077568917688032316821422841386089000118255616	16
110505846047229752326589604332033164031261490444738367057821696	17
48625300818046086265069626850949609369576855533139078535970816	18
13091029273263406516278448784433346603050157320365024326189056	19
2051807629986344848604426466312313602882536535750828575686656	20
172668341932654909847730916643828895856317289938872587780096	21
6779920594885825855179656688221144580134825001572552933376	22
93513608341387693753065554236646869268948208302577156096	23
190169564657928428175235445073924928592047775873499136	24

n = 50

562949953421312	0
1584563250285146014382523678 72	1
25258786038593913696314420232267497472	2
111497663883528088052493346139684549553 02912	3
1950504732106069185498880743102663622557349 96992	4
43986808908422163991671271008158222395355016934 1952	5
2372304453215772293027612525108296940989636353846 14912	6
43708883782226899778992073534426649993574617267371507712	7
342896336296572312788626452396602933626178648026670904 1152	8
1321879126687399725089717108183431927328992859143344037 56032	9
27592527331925580350046844197895383565539052985690094273 82272	10
33360641258284651418295424561582333597274342270934355418284032	11
244943898808782401206748588279813905549188228466934214696108032	12
1128626809755277010891001154691710409069407482076342405218435072	13
3335187589209485410605754478100584503869313357085285891495165952	14
6400561251856035561971453676766502166086271617020900835015524352	15
8010983183865690262955066473679609285693589899424785797283315712	16
6513069084362763436998328438961710974268179118188005215973670912	17
3394843275746967810644900280984549452481581376485622123176394752	18
1107144780747089295831973473588729176556928559212269915479212032	19
217086213192060548804170399429450637351568107705185043440730112	20
24011868084927574822135572062826980240738253043317022467817472	21
1347886398960677448424649707521146025824105689705572961615872	22
31608335579929912720865773356243628045448814004813915226112	23
196535694915671808914892880726989984967498805398829268992	24